Lesson Plan 6 on Angles and Orthogonality

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Lesson Plan 6 defines angles between vectors in n-space, measures them and then uses the standard matrix representation of rotations to prove the addition formulas for sine and cosine via matrix theory. Orthogonality of vectors is defined via the vector dot product and orthogonal matrices are introduced.
This Lesson Plan is widely extended to simulate a live teacher - student interaction.

(version August 5, 2023) iii + 20 p.; 3 - 4 class days

Concepts, Notions and Definitions in Lesson Plan 6

Angles in Space

Teamwork

Plane

Trigonometry

Unit Vector

Unit Circle

Normalized Vector

Linear Function

(continued)

Planar Rotation

Standard Matrix Representation

Trig Addition Formulas

Dot Product Angle Formula

Perpendicular Vectors

Matrix Transpose

Orthogonal Matrix

What determines angles in space and how can we measure them in $\mathbb{R}^n$?

The concept of **angles in space** and their measurement offer a chance for deeper **conceptual understandings** of vectors and matrices on a concrete level.

Here my **startup questions in class** typically are:

What is an angle in $\mathbb{R}^n$?

How does it come about?

What geometric objects of $\mathbb{R}^n$ define an angle?

I then step out, walk the hallways for a minute or two or four while the students deal with this mental task.

Students are often best left alone deliberately for a short interval to discuss, develop and perform this labor by themselves through teamwork.

Occasionally I enter back in to learn about their progress, to give pointers or ask further questions until they are sufficiently clear about . . . .

Questions naturally lead to **group discussions** in class. Students will confer and discuss.

My questions may generate practical student reflections such as:

Take 3 points in n-space, say the origin O, A, and B, or two lines that intersect at O, or ... .

Then maybe:

Look at the plane spanned by O, A, and $B \in \mathbb{R}^n$, or by the two lines through the origin O ... . And :

Make or call origin O the vertex of the angle.

. . . . Or not ... . ; each class is different.

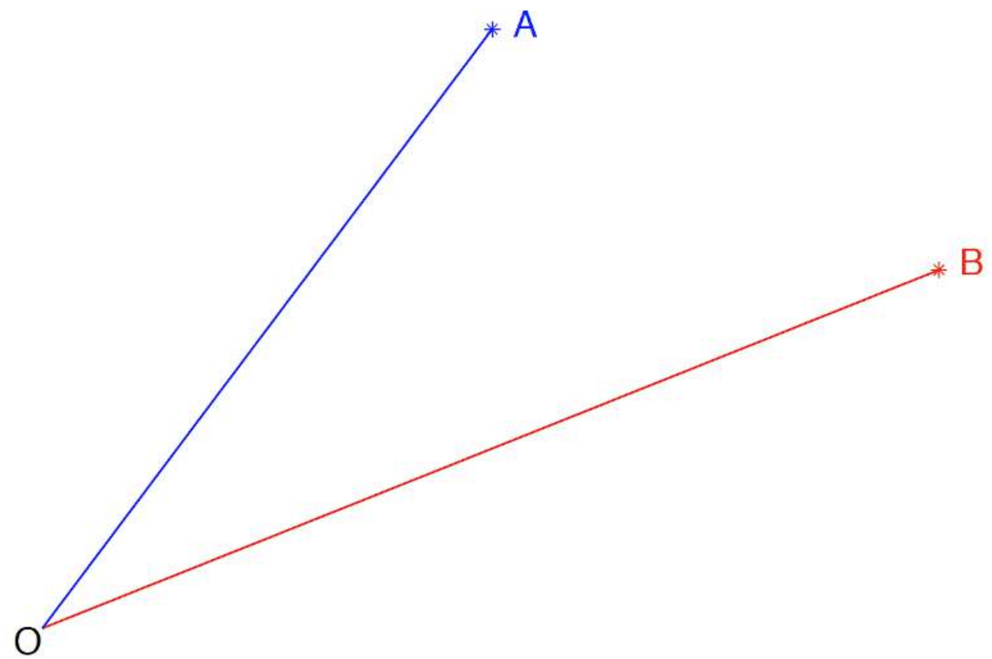
And finally :

Move, rotate and tilt the plane to make it coincide with $\mathbb{R}^2$. And then :

Draw the three points out on paper (or the two intersecting lines), and there it is:

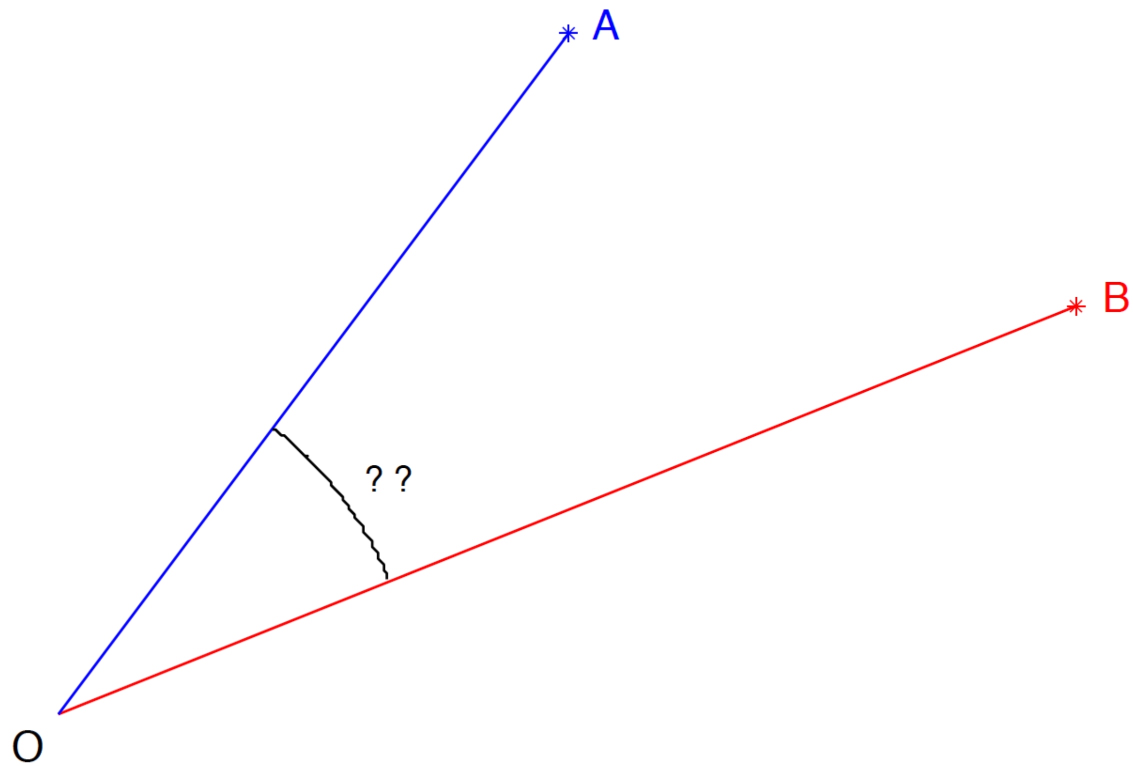
the **angle between** OA **and** OB *(or between the two lines).*

The students can generally construct the **equivalence** between a space angle and its planar representation in $\mathbb{R}^2$.



To visualize that a general plane in $\mathbb{R}^n$ is like our 2-D drawing on paper involves will forces and mental abstraction of a high degree.
This takes students a good amount of time.

Then we use geometry and trigonometry of $\mathbb{R}^2$ for the angle between the rays OA and OB to **measure angles**.



Further questions:

How do we **measure planar angles***?*

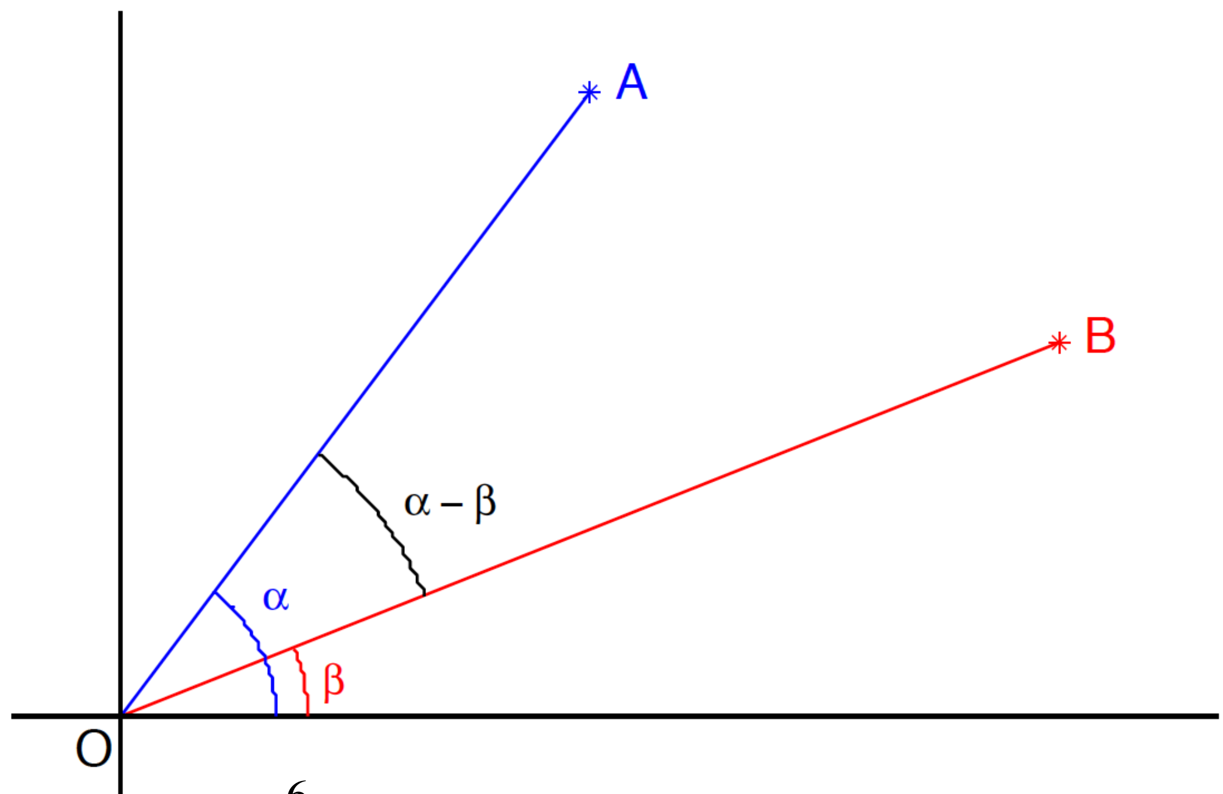
Does anyone remember **trigonometry***?*

What can trig functions do for us here?

How are the elementary trig functions defined?

Then I draw **the standard coordinate axes** into the plot. But to what good ?

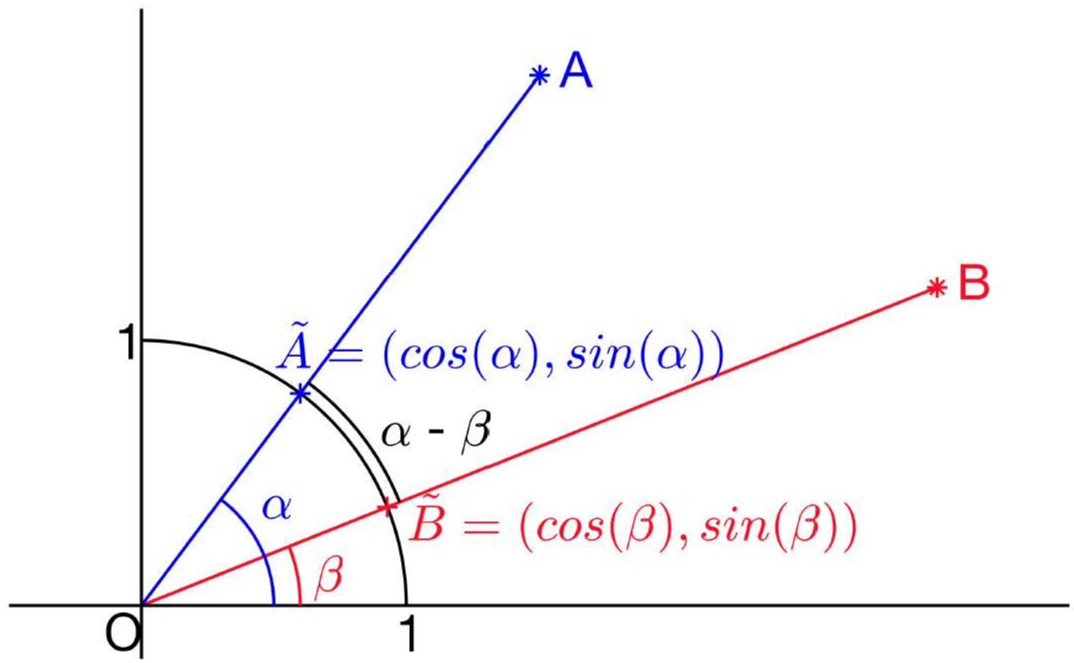
After drawing the coordinate axes and labeling the respective angles, ... *then what ?*



Trigonometry ? *Help ?* *HELP* **!**

From the drawing students may recognize a possible role for **sine** and/or **cosine** here.

The student reflections may lead to the **unit circle** and its intersections $+$ and $+$ with the two rays from O to A and to B :



Students might not correlate the points marked by $+$ and $+$ on the unit circle to the given points A and B

... or correlate to the rays OA and OB.

I, the teacher, must be patient and wait until . . . . .

until some students note that the two $(\cos .., \sin ..)$ points marked by + and + represent the **normalized vectors** from O to A and from O to B of length 1.

Thus as vectors

$$+ = (\cos(\alpha), \sin(\alpha)) = \frac{A}{\|A\|} = \tilde{A} \qquad \text{and}$$

$$+ = (\cos(\beta), \sin(\beta)) = \frac{B}{\|B\|} = \tilde{B}\ .$$

We still need find a **measure for the angle** $\alpha - \beta$ somehow between the rays OA and OB.

What is $\cos(\alpha - \beta)$ *given the coordinates of*

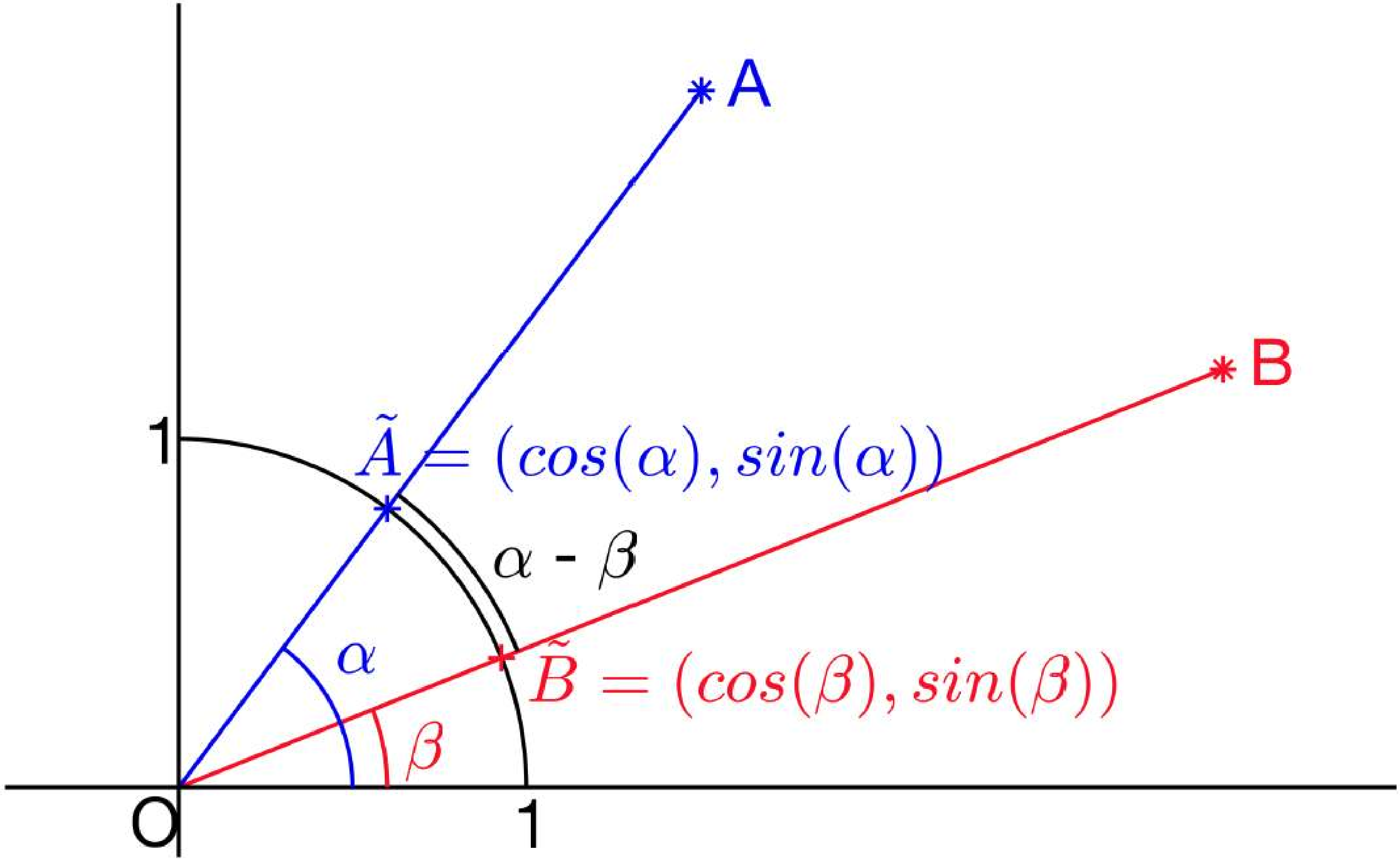
$$\tilde{A} = (\cos(\alpha), \sin(\alpha)), \quad \tilde{B} = (\cos(\beta), sin(\beta))?$$

Can anyone recall the 'addition formulas' for trig functions? Were they ever proved?

Who remembers now?

Can **Linear Algebra** help us solve this geometric problem?

What can we do with $\cos(\alpha - \beta) = \; \ldots.$ *using* **matrices** *to measure the angle between* OA *and* OB*?* Is there a way, what is the way?



The **trigonometric identities** can indeed be established via **linear transformations** and **their standard matrix representations** :

Recall the **linearity condition** of linear functions f

$$f(au + bv) = af(u) + bf(v).$$

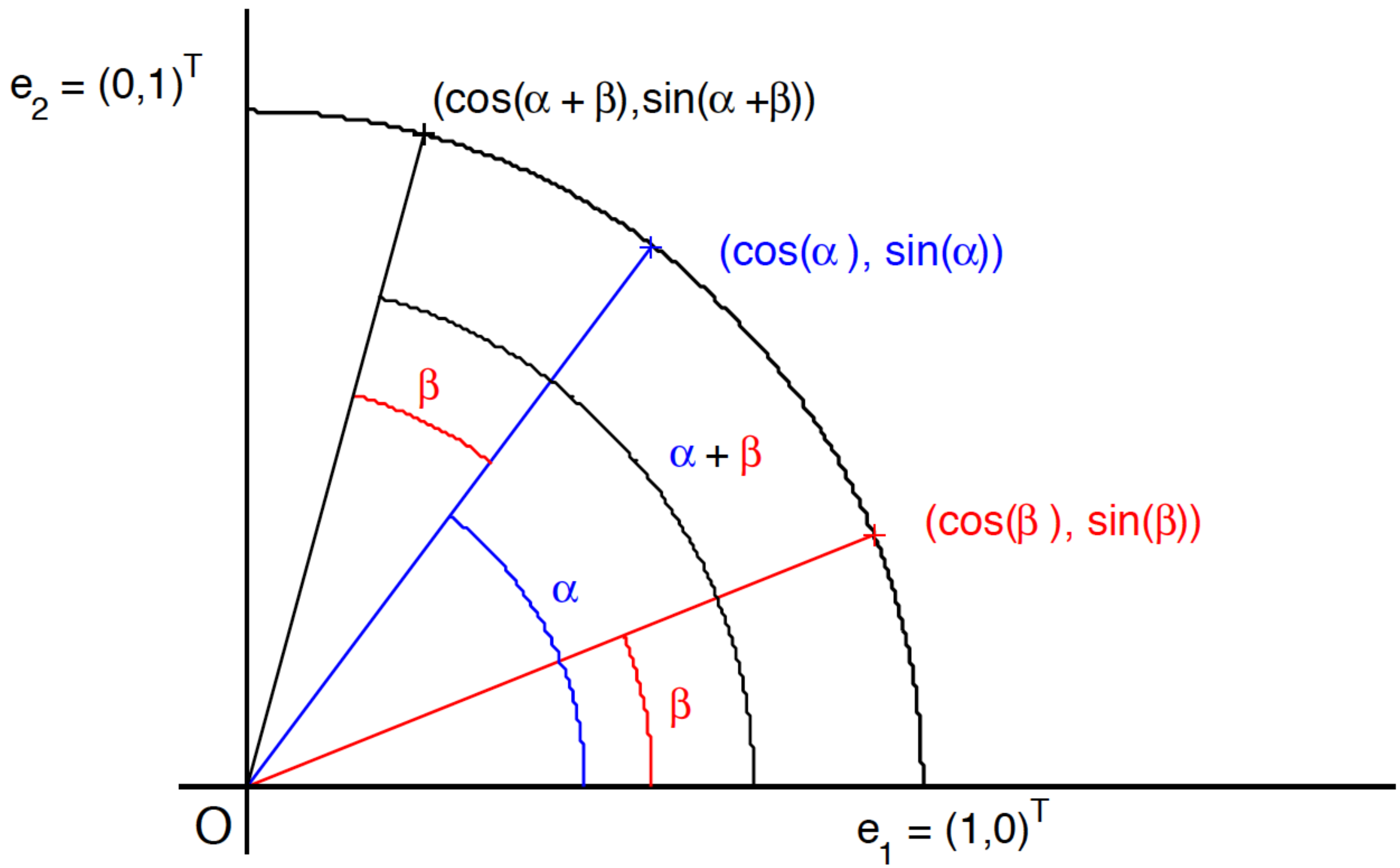
Consider two nonzero vectors au and $bv \in \mathbb{R}^2$ with $a, b \in \mathbb{R}$ and $u, v \in \mathbb{R}^2$ and the diagonal $au + bv$ of the **parallelogram** that they form.

How does a **planar rotation** R_β *around the origin by* β *change this or any* **parallelogram***?*

The rotated sides and their diagonal form another parallelogram that is congruent to the original one, and therefore

$R_\beta(au + bv) = R_\beta(au) + R_\beta(bv)$, i.e., R_β is linear.

Thus the rotation R_β of $\mathbb{R}^2$ around the O is a linear function and has a standard 2 by 2 matrix representation.



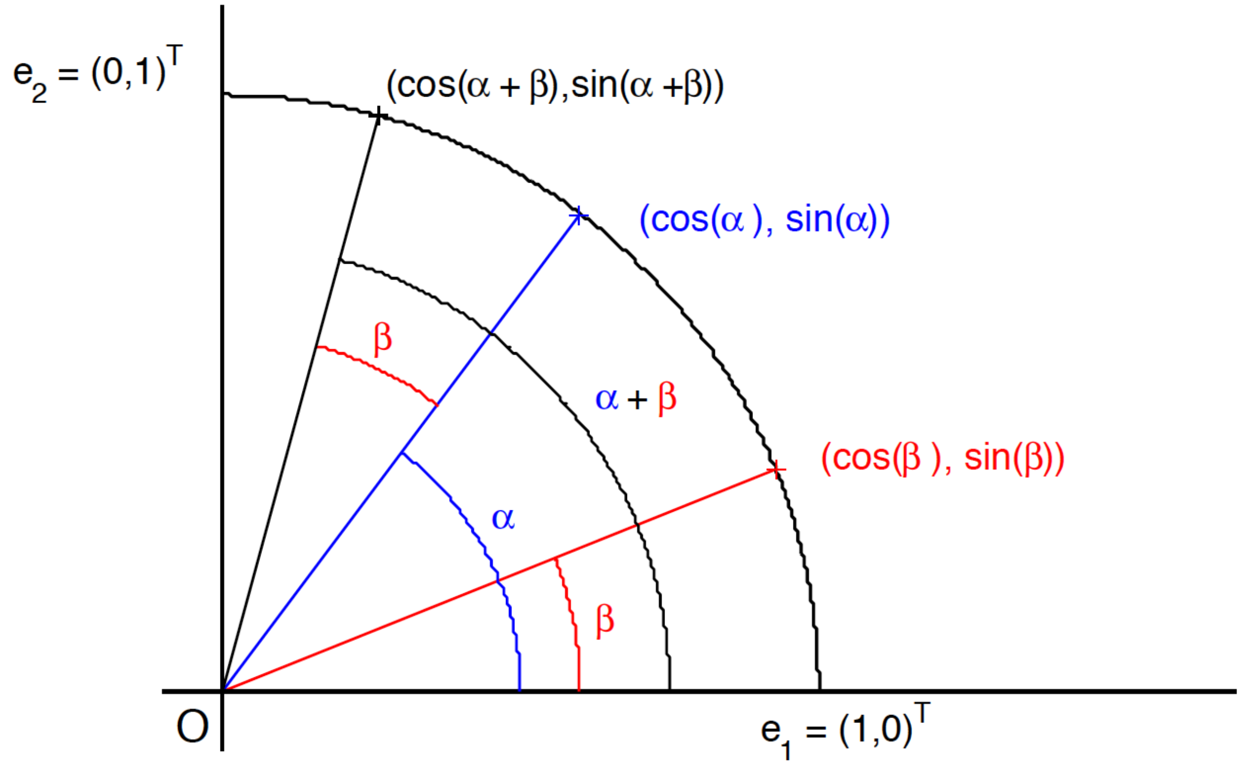
What is the **standard matrix representation** *for the counter-clockwise rotation R_β of the plane around the origin 0 by the angle β?*

The **standard matrix representation** of a **linear transformation** $\mathbb{R}^2 \to \mathbb{R}^2$ contains the images of the **standard unit vectors** $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ of $\mathbb{R}^2$ in its columns.

Thus the **matrix** for the counter-clockwise rotation R_β is

$$R_\beta = \begin{pmatrix} \vdots & \vdots \\ R_\beta \begin{pmatrix} 1 \\ 0 \end{pmatrix} & R_\beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \vdots & \vdots \end{pmatrix}_{2,2} .$$

What are the images of the unit vectors e_1, e_2 of $\mathbb{R}^2$ under counterclockwise rotation by β?



By inspection, $R_\beta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\beta) \\ \sin(\beta) \end{pmatrix}$ and $R_\beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin(\beta) \\ \cos(\beta) \end{pmatrix}$; *Why is ↓ this correct? Explain.*

i.e., $$R_\beta = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix},$$ a **2 by 2 matrix**.

If we now **rotate** the point $(\cos(\alpha), \sin(\alpha))$ by an additional angle β counterclockwise around the origin, it moves to $(\cos(\alpha+\beta), \sin(\alpha+\beta))$.

How does the 2 by 2 rotation matrix R_β map $(\cos(\alpha), \sin(\alpha))^T$*?*

$$\begin{pmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{pmatrix} \overset{!}{=} R_\beta \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\ \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta) \end{pmatrix} \in \mathbb{R}^2 .$$

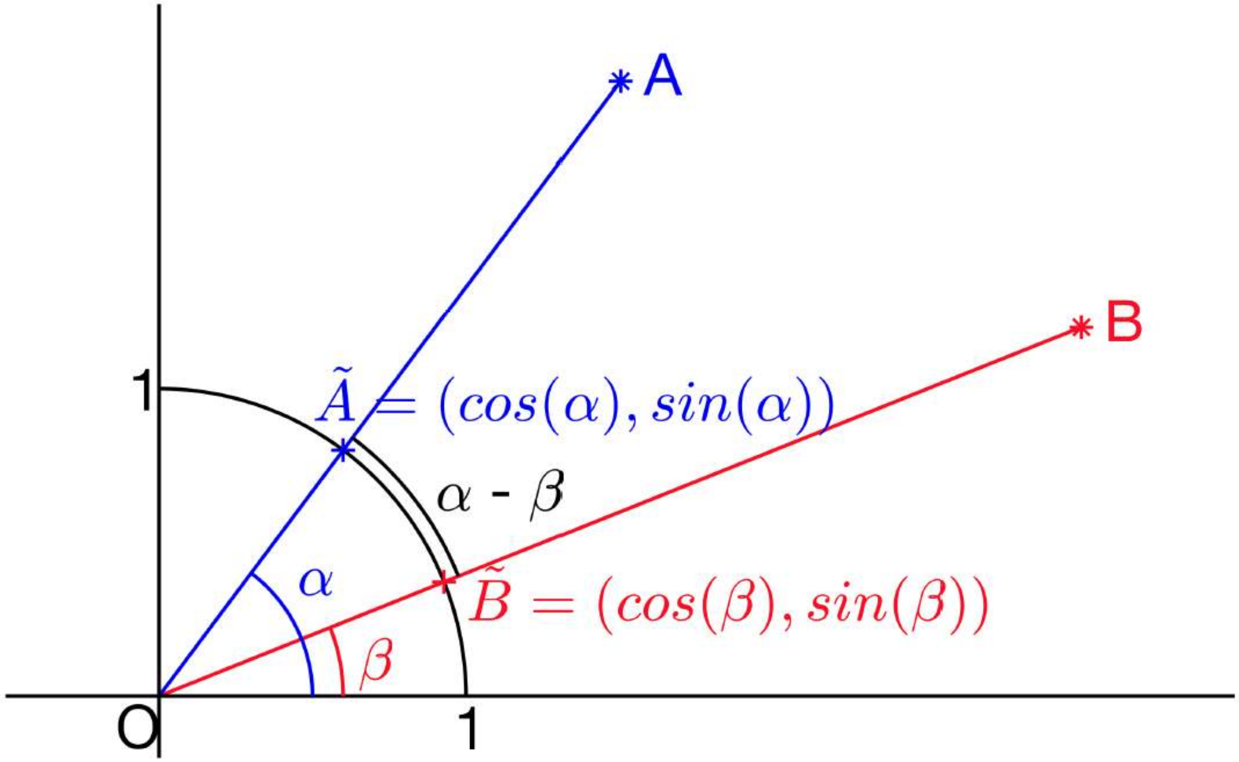
By comparing entries :

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \quad \text{and}$$

$$\sin(\alpha+\beta) = \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta) .$$

Thus we have derived both **trigonometric addition formulas** (for sine and cosine) in one step by using **fundamental concepts** from **Linear Algebra**.

What about $\cos(\alpha - \beta) = \cos(\angle(AOB))$ . . . . . . , *however ?*



Cosine is an **even** function $\cos(-\gamma) = \cos(\gamma)$ and **sine** is **odd**, i.e., $\sin(-\gamma) = -\sin(\gamma)$. Thus

$$\begin{aligned} \cos(\angle(AOB)) = \cos(\alpha - \beta) &= \cos(\alpha + (-\beta)) \\ &= \cos(\alpha)\ \cos(-\beta) - \sin(\alpha)\ \sin(-\beta) \\ &= \cos(\alpha)\ \cos(\beta) + \sin(\alpha)\ \sin(\beta)\ . \end{aligned}$$

$\uparrow$ even fctn $\qquad \uparrow$ odd fctn

What does the cosine difference formula have to do with our original angle problem?

The *geometric angle* is determined by the vectors or line segments OA and OB in $\mathbb{R}^n$.

How does $\cos(\alpha - \beta)$ *relate to the geometry and the coordinates of* A*,* B *and* O*?*

Thus far, we know **three formulas** that relate **space angles** and **vectors** :

$$(1)\ \cos(\angle(AOB)) = \cos(\alpha - \beta) = \cos(\alpha)\ \cos(\beta) + \sin(\alpha)\ \sin(\beta),$$

$$(2)\ (\cos(\alpha), \sin(\alpha)) = \frac{A}{\|A\|} \quad \text{and} \quad (3)\ (\cos(\beta), \sin(\beta)) = \frac{B}{\|B\|}\ .s$$

What are their relationships? How can we express

$$\cos(\alpha)\ \cos(\beta) + \sin(\alpha)\ \sin(\beta) \qquad \textit{or} \qquad \cos(\alpha - \beta)$$

in terms of the given vectors from O *to* A *and* B*?*

The right hand side $\cos(\alpha)\ \cos(\beta) + \sin(\alpha)\ \sin(\beta)$ in equation (1) of the cosine difference formula is the **dot product** of the two **unit vectors** $(\cos(\alpha), \sin(\alpha)) \cdot (\cos(\beta), \sin(\beta))$ derived from OA and OB. Thus

$$\begin{aligned}
\underline{\underline{\cos(\angle(AOB))}} &= \cos(\alpha - \beta) = \cos(\alpha)\ \cos(\beta)\ +\ \sin(\alpha)\ \sin(\beta) && (1)\\
&= (\cos(\alpha), \sin(\alpha)) \cdot (\cos(\beta), \sin(\beta)) && (2),\ (3)\\
&= \frac{A}{\|A\|} \cdot \frac{B}{\|B\|} = \underline{\underline{\frac{A \cdot B}{\|A\|\ \|B\|}}}\ .
\end{aligned}$$

This is the **’dot product’ cosine angle formula for $\mathbb{R}^n$**.

The space angle example takes about 1 or 2 hours of intense class time and student work.

Student question: When is the angle $\angle(AOB)$ a right angle, i.e., when are the rays from O to A and O to B perpendicular?

Precisely when $\cos(\angle(AOB)) = 0$ *or when the dot product* $A \cdot B = 0$ *according to the dot product cosine angle formula.*

We can now solve a typical problem of linear algebra: Given two nonzero vectors x and y in n-space, orthogonalize y with respect to x so that the resulting vector z and x are perpendicular to each other.

To simplify, normalize the vector $x \neq o \in \mathbb{R}^n$ to become a unit vector $\hat{x}$ of length 1 by dividing x by its euclidean length or norm $\|x\| = \sqrt{x \cdot x}$, i.e., $\hat{x} = x/\|x\| = x/\sum x_j^2 \neq o_n$.

Next we set $z = y - (\hat{x} \cdot y)\hat{x}$ and verify that $\hat{x} \perp z$, namely

$$\hat{x} \cdot z = \hat{x} \cdot y - (\hat{x} \cdot y)\hat{x} \cdot \hat{x}) = \hat{x} \cdot y - \hat{x} \cdot y = o_n.$$

If we normalize z to become $\hat{z}$ with $\|\hat{z}\| = 1$, then both vectors $\hat{x}$ and $\hat{z}$ have length 1 and $\hat{x}$ and $\hat{y}$ are mutually orthogonal.

For the n by 2 matrix

$$U = \begin{pmatrix} \vdots & \vdots \\ \hat{x} & \hat{z} \\ \vdots & \vdots \end{pmatrix}_{n,2}$$

note that $U^T_{2,n} \cdot U_{n,2} = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ due to the construction of the vectors $\hat{x}$ and $\hat{z}$.

Here I_2 is the 2 by 2 identity matrix and U^T with 2 rows and n columns denotes the transpose matrix of $U_{n,2}$ that contains the columns of U in its rows.

Vector set orthogonalizing processes and orthogonal matrices $U_{n,n}$ with $U^T \cdot U = I_n$ play a central role in Matlab and in the additional matrix eigenvalue codes for real and complex square matrices that we shall study in Lesson Plan 7 next.