Lesson Plan 2
“Row Reduction” **and** *“Systems of Linear Equations”*

Frank Uhlig

Department of Mathematics and Statistics,
Auburn University, Auburn, AL 36849-5310, USA
uhligfd@auburn.edu

Lesson Plan 2 deals with systems of linear equations and introduces row reduction to find pivots and free columns for a matrix that represents a linear problem. Row reduction is practiced at first by hand and using pencil on paper in order to learn this essential linear algebraic process deeply. For teachers we explain a simple way to create integer model problems that can be row reduced and solved with integer arithmetic. Once the reduction process is well understood and practiced, row reduction is dealt with through computer software and the computed results are interpreted.

(version August 3, 2023) iii + 29 p.; 4 - 6 class days

Row Reduction for Linear Equations

Model Equation

System of Linear Equations

Matrices (Row Operations)

Row Reduction

Pivot

Row Echelon Form

Free Variable

Pencil and Paper Computations

Using Software

Row Reduction of Matrices

Page ii above lists the concepts and definitions of ancient methods to solve Systems of Linear Equations.

Done by hand on pencil and paper, this is a very tedious and error prone process, that will re-occur throughout all our Lessons and it must be understood deeply by our students to be able to understand linear algebraic processes concretely.

Is is also good for students to learn to use software such as Matlab's `rref.m` function to verify their computations and to use `rrefmovie.m` to see how it is done step by step.

But it is still mandatory that students learn how to proceed with row reductions flawlessly themselves on paper.

Goals :

Contents-wise :

Explore and understand mankind's first matrix models : the 5,000 or 6,000 + years old matrix model for systems of linear equations.
Learn to compute row-reductions of matrices as first introduced in Babylonian times many millennia ago in the fertile plains of the Euphrates and Tigris rivers in today's Iraq.

Cuneiform tablet (from Yale U) with Babylonian methods for solving a system of two linear equations.

Pedagogy-wise :

Teach students the ubiquitous duality of Linear Algebra and Matrix Theory:

First we need to abstract a 'verbally given' set of equations that measure tangible real world quantities and transform these equations into a matrix model that carries nothing but numbers,
then perform concrete computing steps to find the linear system's solution and
finally reinterpret the computed result in real world terms again.

Here we need to emphasize and practice the rigorous, mechanical process of 'row-reduction'.

This process is fundamental to Linear Algebra and Matrix Theory.
It is very computer like, tedious and challenging in its 'one way' approach :

to do it right and **do not mess up**.

We start with a classical model that may actually be explained (in Cunei-script) on the Yale Museum clay block:

Story : A child buys 2 apples and 3 oranges one day for 15 silverlings. It comes back the next day and returns one of the apples because it is rotten and instead it buys two more oranges, paying 3 silverlings in this exchange.
What is the price of one apple and of one orange?

Setup : First we need to construct model equations for these transactions.
On the first day, the price of 2 apples and 3 oranges is 15 silverings or

$$2x + 3y = 15s \tag{1}$$

where x is the price of one apple and y is that of one orange.

On the second day the refund for 1 apple and buying 2 oranges instead required 3 silverings in pay or

$$-x + 2y = 3s\ . \tag{2}$$

Data matrix :

	x apples	y oranges		silverlings
day 1	2	3	=	15
day 2	– 1	2	=	3

How can we transform this data matrix and formulas (1) and (2) into a matrix $\times$ vector equation?

On each day the child bought x apples and y oranges, so the 'produce vector' is $\begin{pmatrix} x \\ y \end{pmatrix}$ and the silverlings vector is $\begin{pmatrix} 15 \\ 3 \end{pmatrix}$.

And

$$\begin{pmatrix} 2x + 5y \\ -x + 2y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} x + \begin{pmatrix} 3 \\ 2 \end{pmatrix} y = \begin{pmatrix} 15 \\ 3 \end{pmatrix}\ . \tag{3}$$

Recall that multiplying $A_{m,n}$ by an n-vector x on the right creates a linear combination of the column vectors in A.

Using duality and reading $A_{m,n} \cdot x = \begin{pmatrix} \vdots & & \vdots \\ c_1 & \cdots & c_n \\ \vdots & & \vdots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$= \begin{pmatrix} \vdots \\ c_1 \\ \vdots \end{pmatrix} x_1 + \begin{pmatrix} \vdots \\ c_2 \\ \vdots \end{pmatrix} x_2 + \ldots + \begin{pmatrix} \vdots \\ c_n \\ \vdots \end{pmatrix} x_n$ backwards gave

us formula (3) in matrix $\times$ vector form as

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} x + \begin{pmatrix} 3 \\ 2 \end{pmatrix} y = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 3 \end{pmatrix}. \quad (3)$$

With $A_{2,2} = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$, $\tilde{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $b = \begin{pmatrix} 15 \\ 3 \end{pmatrix}$, our linear equation model now has the standard form $A \cdot \tilde{x} = b$.

How can we solve an equation in the form $A \cdot x = b$ for a compatible set of a matrix $A_{m,n}$, a vector $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$? How did the Babylonians solve systems of linear equations?

What operations on sets of equations are legal to perform without affecting their solution x?

(A) We can add and subtract multiples of one equation to or from another.

(B) We can change the order of the equations and reshuffle them at will.

(C) We can multiply both sides of any equation by a non-zero factor.

And the solution(s) will stay the same.

Row reduction of augmented matrices $(A|b)_{m,n+1}$ using these operations were one method in Babylon to solve linear systems $Ax = b$. We shall do the same.

To solve $Ax = b$ we assemble the data in $A_{m,n}$ and $b \in \mathbb{R}^m$:

$$
\begin{array}{cccc|c}
\multicolumn{4}{c|}{A_{m,n}} & b \\
\hline
a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\
a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_m
\end{array}
$$

Next we append the data matrix by an 'operations' column on the right where we will detail the row operations that zero out certain column entries.

$$
\begin{array}{cccc|c||l}
\multicolumn{4}{c|}{A_{m,n}} & b & \text{row operations} \\
\hline
a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 & \\
a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 & \\
\vdots & \vdots & & \vdots & \vdots & \\
a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_m &
\end{array}
$$

The aim of row reduction is to zero out the lower triangle of the matrix $A_{m,n}$. If $a_{1,1}$ is non-zero, we can use it as a $\boxed{\text{pivot}}$ to zero out the lower placed entries of column 1 by subtracting proper multiples of row 1 from the lower rows 2, ..., m in $(A|b)$.

$$
\begin{array}{cccc|c||c}
 & A_{m,n} & & & b & \text{row operations} \\
\hline
\boxed{a_{1,1}} & a_{1,2} & \cdots & a_{1,n} & b_1 & (\text{ if } a_{1,1} \neq 0\ ) \\
a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 & -\ a_{2,1} \ / \ \boxed{a_{1,1}} \cdot \text{row } 1 \\
\vdots & \vdots & & \vdots & \vdots & \vdots \\
a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_m & -\ a_{m,1} / \ \boxed{a_{1,1}} \cdot \text{row } 1
\end{array}
$$

Recall the zero rule of mathematics :

(ZERO rule) One must **not divide by zero**; ever !

When performing row operations on the augmented matrix $(A|b)$ we have to update the entries below and to the right of the pivot, including those in b.

Zeroing of the entries in column 1 below the pivot gives us the updated $(\tilde{A}|\tilde{b})$ data on the left of the vertical double || line :

$A_{m,n}$				b	row operations
$\boxed{a_{1,1}}$	$a_{1,2}$	$\cdots$	$a_{1,n}$	b_1	
$a_{2,1}$	$a_{2,2}$	$\cdots$	$a_{2,n}$	b_2	- $a_{2,1}/a_{1,1} \cdot$ row 1
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$a_{m,1}$	$a_{m,2}$	$\cdots$	$a_{m,n}$	b_m	- $a_{m,1}/a_{1,1} \cdot$ row 1
$a_{1,1}$	$a_{1,2}$	$\cdots$	$a_{1,n}$	b_1	
0	$\boxed{\tilde{a}_{2,2}}$	$\cdots$	$\tilde{a}_{2,n}$	$\tilde{b}_2$	(if $\tilde{a}_{2,2} \neq 0$)
0	$\tilde{a}_{3,2}$	$\cdots$	$\tilde{a}_{3,n}$	$\tilde{b}_3$	- $\tilde{a}_{3,2}$ / $\boxed{\tilde{a}_{2,2}}$ $\cdot$ updated row 2
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
0	$\tilde{a}_{2,2}$	$\cdots$	$\tilde{a}_{m,n}$	$\tilde{b}_m$	- $\tilde{a}_{m,2}$ / $\boxed{\tilde{a}_{2,2}}$ $\cdot$ updated row 2
and so forth with $\boxed{\hat{a}_{3,3}}$					updating rows 4 to min$(m-1, n)$

and continuing until column number min$(m-1, n)$ is reached.

This process is called row reduction and its final upper triangular matrix is row echelon form of the original augmented matrix $(A|b)$.

What can we do if a designated pivot spot $a_{j,k}$ is zero when trying to eliminate the entries in column k below row j?
[Remember : *the row index j in $a_{j,k}$ precedes the column index k; always row before column.*]

Thus far we have not used rule (B) to swap rows or equations that does not affect the solution of a system of linear equations.

If there is a non-zero entry $a_{i,k}$ in column k with row index $i > j$ when $a_{j,k} = 0$, interchange rows i and j and eliminate all entries of the updated augmented matrix $(\tilde{A}|\tilde{b})$ in column k below row j.

If all entries in columns k below row j are zero, move to the next column $k+1$ and try to move a non-zero pivot into position $j, k+1$. If impossible, move to the right to column $k+2$ and so on and on.

We want to solve sets of linear equations $A_{m,n}x = b$ from a row echelon form $(\tilde{A} \mid \tilde{b})$ of the augmented matrix $(A|b)_{m,n+1}$.

Here is a generic row echelon form for a 5 by 7 matrix A and a right-hand side b in 5-space that 'boxes' the pivots and labels every entry with a 0 or a $*$ if non-zero.

$$(\tilde{A} \mid \tilde{b}) = \left(\begin{array}{ccccccc|c} \boxed{*} & * & * & * & * & * & 0 & * \\ 0 & \boxed{*} & * & * & * & * & * & * \\ 0 & 0 & 0 & \boxed{*} & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & \boxed{*} & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{array}\right) \tag{5}$$

Clearly the solution x of $Ax = b$ and $\tilde{A}x = \tilde{b}$ coincide since row additions and row scaling or interchanges do not affect the solution of a linear system. Is (5) solvable and how is it done?

Let us investigate the leading 3 by 3 block of the symbolic row echelon form in (5) as if it were the row echelon form of a 3 by 2 matrix B and a right hand side c in the linear system $By = c$

$$(\tilde{B} \mid \tilde{c}) = \left(\begin{array}{cc|c} \boxed{*} & * & * \\ 0 & \boxed{*} & * \\ 0 & 0 & 0 \end{array} \right) . \qquad (6)$$

Here the unknown solution y has two components. *Why?*

Under what circumstances can this system be solved?

Is the solution unique or are there multiple solutions?

Clearly the second row in (6) determines y_2 uniquely, while the first row does so for y_1, whatever the values of $*$ are.

Note that boxed pivots $\boxed{*}$ are always assumed to be non-zero.

Next take the leading 3 by 4 block of (5) to describe the row echelon form of a system matrix $C_{3,3}$ and the right-hand side vector $d \in \mathbb{R}^3$ in

$$(\tilde{C} \mid \tilde{d}) = \left(\begin{array}{ccc|c} \boxed{*} & * & * & * \\ 0 & \boxed{*} & * & * \\ 0 & 0 & 0 & \rlap{0}{*} \end{array}\right) . \tag{7}$$

Here the unknown solution z has three components.
Under what circumstances can this system be solved?
Solved uniquely?
The system cannot be solved at all if the last entry $\rlap{0}{*}$ of $\tilde{d}$ is nonzero since $y_1 \cdot 0 + y_2 \cdot 0 + y_3 \cdot 0\ (= 0)$ cannot be non-zero.
For any value assigned to z_3, the system with $\tilde{d}_3 = 0$, however, can be solved and there are infinitely many solutions here.

We are free to choose z_3 arbitrarily and then solve for z_2 and z_1 in turn.

Finally we consider the leading 4 by 5 block of (5) and describe the row echelon form of a system matrix $D_{4,5}$ with right-hand side vector $g \in \mathbb{R}^4$ in

$$(\tilde{D} \mid \tilde{g}) = \left(\begin{array}{ccccc|c} \boxed{*} & * & * & * & * & * \\ 0 & \boxed{*} & * & * & * & * \\ 0 & 0 & 0 & \boxed{*} & 0 & * \\ 0 & 0 & 0 & 0 & \boxed{*} & 0 \end{array}\right). \tag{8}$$

Here the unknown w has five components.
Under what circumstances can this system be solved?
Solved uniquely?

The system can always be solved.

For any value assigned to g_3, the system can be solved and there are infinitely many solutions here since the third component of the solution can be chosen arbtrarily.

How about the row exchelon form of (A, b) in equation (5)?

$$(\tilde{A} \mid \tilde{b}) = \left(\begin{array}{ccccccc|c} \boxed{*} & * & * & * & * & * & 0 & * \\ 0 & \boxed{*} & * & * & * & * & * & * \\ 0 & 0 & 0 & \boxed{*} & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & \boxed{*} & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \tag{5}$$

How many components does the the unknown solution x of $Ax = b$ have?

Under what circumstances can this system be solved? Solved uniquely?

Student answers please, ...

A historical aside :

Our original 'child buying fruit' model

$$\begin{aligned} 2x + 3y &= 15s \quad (1) \\ -x + 2y &= 3s \quad (2) \end{aligned}$$

can be interpreted geometrically (for $s = 1$ unit) as a set of two line equations in x and y whose intersection we want to find.

Equation (1) has the slope-intercept formula $y = -2/3\,x + 5$ while equation (2) becomes $y = 1/2\,x + 3/2$.

Question for students : *For which right hand sides of payment data are the respective apple and orange prices realistic, i.e., positive for both fruits?*

Graphical solutions to linear equations were discussed on Babylonian Cuneiforms as well as the Gaussian row reduction method. For linear systems beyond dimension $n = 2$, row reduction is more practical than trying to find, e.g., plane intersections when $n = 3$.

Determining land boundary lines accurately when the spring floods receded around Babylon has given humankind linear equations, row reduction and the thought of matrix computations 6,000 + years ago. These ideas then traveled to Egypt and the Nile delta and on.

Next some ”Questionable Statements” :

(1) All columns of a row echelon form of an augmented matrix $(A|b)$ either have a pivot or the associated solution set has a free variable.

True or false? Which columns are pivot columns, which are free variable columns in formulas (5) through (8)?

(2) Only those linear systems are solvable whose row echelon form's right-hand side has a nonzero entry below the lowest reduced matrix row with a pivot.

True, sometimes true or always false? Explain please.

(3) Is the row echelon form of an augmented matrix $(A|b)$ unique or can there be multiple such row reduction results?

How many? Discuss, please.

The row reduced matrix $\tilde{A}$ of $A_{m,n} = \begin{pmatrix} \cdots & r_1 & \cdots \\ & \vdots & \\ \cdots & r_m & \cdots \end{pmatrix}_{m,n}$

is achieved by adding or subtracting rows r_i of A, by scaling them and maybe by changing their order.

Therefore the rows $\tilde{r}_i$ in the row echelon form $\tilde{A}$ of A are linear combinations of the rows $r_1, ..., r_m$ in the original A, maybe with rows $\tilde{r}_i$ interchanged.

Conceptually thinking backwards, the non-zero rows in $\tilde{A}$ should be able to recreate the original rows r_i of A by judicious linear combinations and conversely, the set of all linear combinations of the row vectors r_i should be identical to the set of all possible linear combinations of the non-zero or pivot rows in $\tilde{A}$.

We shall study this dual concept in more detail later

Regarding the column vectors in $A_{m,n} = \begin{pmatrix} \vdots & & \vdots \\ c_1 & \cdots & c_n \\ \vdots & & \vdots \end{pmatrix}$ and the columns in its row reduced form $\tilde{A}$ is more complicated.

The original columns in A in the position of the pivots in A's row echelon form $\tilde{A}$ suffice to express all linear combinations of the n columns c_i of A.

For example for $A_{5,7}$ in formula (5), the four pivots in $\tilde{A}$ indicate four columns c_1, c_2, c_4, c_5 in A can be used to express all linear combination of the seven columns of A.

All this will become clearer in our next Lesson on subsets of vectors, spanning sets and minimal spanning set of vectors and matrix inversion.

This section's task is to play loosely with row reductions. Students must learn and repeatedly practice the rigorous way to delete or zero entries in an updated column below the pivot and how to deal with columns that do not have a pivot where desired, i.e., deal with free columns and still proceed to a row echelon form R of any rectangular matrix $A_{m,n}$ or of any augmented matrix $(A|b)$.

Here we also treat systems of linear equations intuitively when looking at applications of row echelon form reductions. In the future we will have to fathom what a row echelon form or a row reduction can do for us as regards subspaces and their generating (still undefined and unnamed) 'bases'.

But instructors need to have access now to simple matrix examples whose row echelon forms are easy to compute.

(1) Start from an integer upper staircase form $\tilde{A}_{m,n}$ that is indeed a genuine matrix row echelon form with pivot columns and free columns.
(2) Now reverse the row reduction process by multiplying $\tilde{A}_{m,n}$ on the left by a small entries integer matrix $B_{m,m}$. Then $C = B_{m,m} \cdot \tilde{A}_{m,n}$ is an integer matrix that has the same row and column dimensions m and n as $\tilde{A}_{m,n}$. And one possible row echelon form of C is $\tilde{A}$ which can be easily computed over the integers.

This can generate dozens and dozens of good homework examples for row reductions that cannot be guessed and that require genuine work to compute.

By hand with paper, pencil and eraser, and over the integers.

Teacher's Preparations start with an integer row echelon form:

$$R_{4,6} = \begin{pmatrix} -1 & 2 & 3 & -2 & -1 & -2 \\ 0 & 2 & 1 & 1 & 3 & 2 \\ 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ for example .}$$

Why is R in row echelon form? Is any information (entries and their row and column distribution) unique?

Next the teacher muddles this row echelon form up by replacing the structured rows of R with integer linear combinations of R's rows by left multiplication R with a 4 by 4 matrix such as

$$B = \begin{pmatrix} 2 & 0 & -1 & 4 \\ -1 & 1 & -1 & 3 \\ 1 & 2 & 0 & 2 \\ 3 & -1 & 2 & 1 \end{pmatrix}.$$ This results in the dense matrix

$$S_{4,6} = B \cdot R = \begin{pmatrix} -2 & 4 & 6 & -6 & -3 & -8 \\ 1 & 0 & -2 & 1 & 3 & 0 \\ -1 & 6 & 5 & 0 & 5 & 2 \\ -3 & 4 & 8 & -3 & -4 & 0 \end{pmatrix}.$$

The **Students' Task** *then is to compute the (or a) row echelon form of S using the row modifying scheme on p. 9 in conjunction with the 3 legitimate rules (A), (B) and (C) of row reduction.*

Note that clearly the scheme of fractional row operations from p. 9 is cumbersome. However if we scale each newly found pivot row in S and its updates to have the pivot 1 by using the legit scaling rule (B), then taking quotients becomes unnecessary in the row operations and we can compute the row echelon form of S entirely over the rationals $\mathbb{Q}$. Therefore we replace the top row of S by $\begin{pmatrix} 1 & -2 & -3 & 3 & 1.5 & 4 \end{pmatrix}$.

row reducing the dense $S_{4,6}$						row operations
$\boxed{1}$	-2	-3	3	1.5	4	scaled beforehand
1	0	-2	1	3	0	$-1 \cdot$ row 1
-1	6	5	0	5	2	$+1 \cdot$ row 1
-3	4	8	-3	-4	0	$+3 \cdot$ row 1
$\boxed{1}$	-2	-3	3	1.5	4	
0	$\boxed{2}$	1	-2	1.5	-4	scale row for pivot $\boxed{1}$
0	4	2	3	6.5	6	
0	-2	-1	6	0.5	12	
$\boxed{1}$	-2	-3	3	1.5	4	
0	$\boxed{1}$	0.5	-1	0.75	-2	
0	4	2	3	6.5	6	$-4 \cdot$ row 2
0	-2	-1	6	0.5	12	$+2 \cdot$ row 2

(continued ..)

updated $S_{4,6}$						row operations
$\boxed{1}$	-2	-3	3	1.5	4	
0	$\boxed{1}$	0.5	-1	0.75	-2	
0	0	0	7	3.5	14	scale row 3 for pivot $\boxed{1}$
0	0	0	4	2	8	
$\boxed{1}$	-2	-3	3	1.5	4	
0	$\boxed{1}$	0.5	-1	0.75	-2	
0	0	0	$\boxed{1}$	0.5	2	
0	0	0	4	2	8	$-4 \cdot$ row 4
$\boxed{1}$	-2	-3	3	1.5	4	
0	$\boxed{1}$	0.5	-1	0.75	-2	
0	0	0	$\boxed{1}$	0.5	2	
0	0	0	0	0	0	

Is the final row-wise updated form of the originally dense matrix $S_{4,6}$ a row echelon form for $S = B \cdot R$? *Explain.*

What are the differences between R and S, with $S = B \cdot R$ as originally given, and in the row echelon forms R and row reduced S?

How many row echelon forms can any rectangular matrix have?

How many pivots can any row echelon form of the same dense matrix have? Different numbers? In different positions?

How many free variable columns must a row echelon form of a matrix $A_{m,n}$ have when $m \neq n$, $m = n$, $m > n$, or $m < n$?

Why? Why? Discuss all these questions, in class and privately.

Practice row reduction until flawlessly done every time.

Row reduction of matrices and solving systems of linear equations is one of the easiest and best understood tasks in Numerical Linear Algebra that is easily delegated to software.

Students can check their pencil and paper REF calculations by using Matlab's `rref.m` row echelon form finder, or if wanting to see the row reduction steps step by step, there is the `rrefmovie.m` m-file in Matlab that will do just that.

We encourage students in beginning Linear Algebra classes to familiarize themselves with software such as Matlab, Mathematica, Python or ... – all the while perfecting their personal mental acuity and pencil and paper skills with hand computations.

If there are discrepancies between the computer output and the paper and pencil result, students and teachers must study how many different row echelon forms exist for any problem and how to correctly decide whether two different row echelon forms are in fact equivalent or not.

Outlook : What comes next?

In Lesson Plan 3 we will deal with linear or vector spaces that are defined as vector spans or the range, or the nullspace of a matrix. We will introduce the concepts of linear (in-)dependence of vectors in terms of matrix row echelon form reductions and pivot or free column counts there.

Then we study the composition of linear transformations in terms of matrix multiplication. For square matrices we shall find matrix inverses if they exist and rebrand the row reduction process as an LR matrix factorization.

All this will help us with extracting the very essence of linear transformations, aka matrices, in terms of their eigenvalues and eigenvectors in Lesson Plans 4, 5 and beyond.