Lesson Plan 1
or
an Introduction to
“Linear Algebra” **and** *“Matrix Theory”*

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This Introduction sets up our path into elementary Linear Algebra and Matrix Theory. Starting with vectors and matrices, their interactions and studying linear combinations and the dot product of vectors, we arrive at Linear Transformations. These are defined abstractly via the parallelogram law and then equivalently as matrix times vector products. This equivalence is explicitly proved by using the Riesz Representation Theorem. Same dimensioned vectors and compatibly sized matrices can be added and scalar multiplied and with the Standard Matrix Representation of Linear Transforms, the game is on.

(version August 2, 2023) v + 22 p.; 4 - 5 class days

Linear Algebra $\leftrightarrow$ Matrix Theory

Concepts, Notions and Definitions in Lesson Plan 1

Vectors

in $\mathbb{R}^n$ or $\mathbb{C}^n$

Matrices (rows before columns)

Vector Algebra, Matrix Algebra

Linear Transformations

Dot Product of vectors

Linear Combinations of vectors

Unit Vectors e_i

Standard Basis $\{e_i\}$

Standard Matrix Representation of linear transforms

Matrix $\times$ Vector Product

Linear Algebra $\leftrightarrow$ or $\equiv$ **Matrix Theory ?**

The previous page lists a few fundamental concepts and definitions for an introductory modern Matrix Theory college course.

Not all of these notions and 'words' can become instantly familiar to our students, but they will re-occur and re-occur here and there.

And it would be great if teachers would just revisit them when need be and they arise again.

Our Goals :

Contents-wise *:*

We introduce n-vectors and m by n matrices lightly; study their interaction in light of linear transformations; introduce unit vectors, the vector dot-product, component functions, Riesz Theorem and the standard matrix representation $A_{m,n}$ of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.

We formally prove that linear transformations and standard matrix representations are equivalent and Riesz's Representation Theorem.

We explain the first two formal (and simple) proofs of the course slowly and deliberately. There will be many student question as to why, how , ... here. Practice !

Pedagogy-wise :

We aim to engage the students and with the students on day 1;
to ask questions and accept their answers, to discuss them and guide them;
to ask for examples and simple computations in class and
to deal with misunderstandings in a respectful, open, even a 'loving' ('That's ok ...') way.

Try to elicit student answers to student questions ('How could you, would you explain this to another student down your hall residence...?').
We build the course on student input, be they understandings or misunderstandings.
And we share our knowledge freely and openly and do not ever rush through the syllabus.

A Lesson Plan for the first Class Meeting(s)

"Linear Algebra" and "Matrix Theory"

***Linear Algebra* deals with vector spaces *in the abstract*, while**
***Matrix Theory* deals with vector spaces *concretely*.**

This is how I would start the first class meeting of a first Linear Algebra class :
"Welcome to your first class on Linear Algebra and Matrices.
Here we will study many concepts and methods that may be new for you, so please bear with me if and when not every notion is completely clear at its first mention; talk with your seat neighbor(s), ..., and we will go over these many times. In a few weeks weeks you will become experts, too.
Welcome and let me begin with some words of explanation."

Abstract Linear Algebra is popular as a first introduction into Algebra.

Here we could introduce finite dimensional vector spaces through 11 axioms for vector addition, scalar vector multiplication and their distributive properties linking vector addition and scalar vector stretching.

And then study linear transformations and the structure of sets of vectors.

We would argue through several notions for linear vector (in)dependence when we study bases and subspaces and learn how to construct indirect proofs and proofs by induction.

Such an approach would introduce students into mathematicians' lingo and prepare them to think like (abstract) mathematicians.

This is a worthy endeavor and worth the effort for maths, science and engineering majors.

Matrix Theory instead deals with finite dimensional vector spaces and linear transformations concretely.

A matrix is a rectangular array of numbers, real or complex. This array interacts with vectors as any linear transformation does, but concretely by matrix $\times$ vector multiplication.

Sets of vectors, their linear (in)dependence and possible basis structure can be read off the row echelon form of their column vector matrix as we will see in Lesson Plan 2.

Likewise, the ranges and nullspaces of linear transformations are displayed by the row echelon form of their standard matrix representation.

Eigenvalues and eigenvectors of linear transformations are best computed by matrix eigensolving software.

Our first immediate task, however, is to link Linear Algebra and Matrix Theory through the standard matrix representation of linear transformations.

The basic notions and tools are vectors and their accumulation in rectangular or square matrices, as well as the algebra behind the interaction of vectors and matrices.

The preceding two pages are directed mostly to instructors. But they could serve as a reminder to students of the subjects that this set of Lessons has addressed when the semester ends and exams loom.

Now is also a good place to just stop, contemplate and let students and things settle, maybe scroll back and mention that all of these 'new words' will become 'household items' quickly;

if students just relax and watch how vectors and matrices and related notions are introduced and handled.

Now the technical part of the course begins concretely.

Vectors contain a number of entries that may be real numbers, complex numbers, named variables; such as the column vector

$$x = \begin{pmatrix} 2.35 \\ \pi \\ 5\sqrt{-1} \\ x_4 \\ -70 \\ x_6 \\ -19 \cdot s \end{pmatrix}$$ or the row vector $y = \left(\ 1, -2, 3i, 2^2, t^3, 6 \ \right)$.

x with seven entries lies in 7-dimensional space, y belongs to a 6-dimensional space.

Now students might want to think up some examples of vectors and of non-vectors; play with them: try to add them, multiply them, stretch them by a scalar factor, reverse their direction, find the zero vector ... . Let everyone have fun and the students get chalk on their hands!

To create a matrix we bunch equally dimensioned vectors together, stacking column vectors left to right and row vectors top to bottom:

If $u = \begin{pmatrix} 2.2 & -4 & 56 & 2\sqrt{7} \end{pmatrix}$ and $v = \begin{pmatrix} 2 & 0 & 17 & -88i \end{pmatrix}$ are two 4-dimensional row vectors, then

$$A = \begin{pmatrix} \cdots & u & \cdots \\ \cdots & v & \cdots \end{pmatrix} = \begin{pmatrix} 2.2 & -4 & 56 & 2\sqrt{7} \\ 2 & 0 & 17 & -88i \end{pmatrix}$$

is a 2 by 4 matrix.

Often we write a matrix B with its column and row numbers subscripted as $B_{m,n}$. Thus $A = A_{2,4}$ above.

Always remember *“rows before columns”*, or the number m of rows of a matrix B precedes the number n of columns in **double subscript notation** $B_{m,n}$.

With $m = 2$ and $n = 4$ for A above we have $A_{m,n} = A_{2,4}$. Note that neither $x_{7,1}$ nor $y_{1,6}$ from two pages back can appear in a matrix that has either $u_{1,6}$ or $v_{1,6}$ as a row or column since the vector dimensions (rows before column indices) do not conform.

To study and comprehend vector spaces and their linear transformations and their structures is the task of abstract Linear Algebra.

Matrix Theory **instead** studies the concrete effects of matrices when they map vectors from one space to itself or to another.

Let this distinction set in.

Then ask and listen to the students' future uses of Linear Algebra and/or of Matrix Theory ...

We have to answer two questions now:

Question 1: What are linear transformations of vector spaces?

Question 2: How do matrices map vectors, concretely and practically?

DEFINITION

A function f between two vector spaces $\mathcal{U}$ and $\mathcal{V}$ is a linear transformation if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \qquad (*)$$

for all vectors $x, y \in \mathcal{U}$ and all scalars α and β.

In equation $(*)$ the expression $\alpha x + \beta y$ is the diagonal of the parallelogram with sides αx and βy.

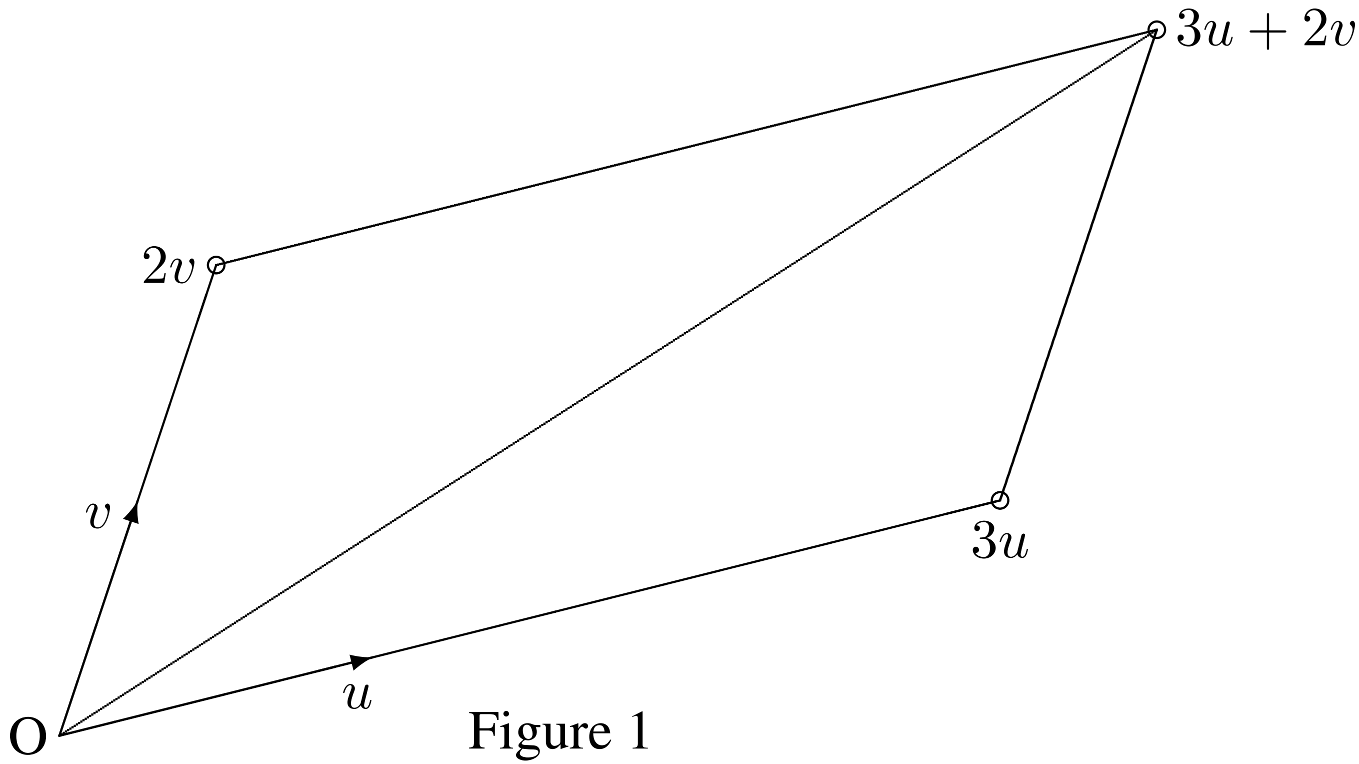

Figure 1

In equation $(*)$, the right hand expression $\alpha f(x) + \beta f(y)$ is the diagonal of the parallelogram in the range space with sides $\alpha f(x)$ and $\beta f(y)$.

Linear functions preserve the parallelogram law between their domain and range spaces.

Think of f as a delivery service that sends α items x and β items y in one package or it sends α packages with x inside and β packages with y inside. The effect will be the same if f is linear. never mind the extra packaging material ...

Any two real or complex entry vectors $x = (x_i)$ and $y = (y_i)$ in n-space can be added by adding their components become $x + y = (x_i + y_i)_n$.
Likewise vectors can be multiplied by any scalar α to become $\alpha x = (\alpha x_i)_n$.

This describes the linear algebraic vector structure of n-space.

Likewise compatibly sized real or complex matrices $A_{m,n} = (a_{i,j})$ and $B_{m,n} = (b_{i,j})$ are added entry by entry to become $A + B = (a_{i,j} + b_{i,j})_{m,n}$.
Single matrices $A_{m,n}$ are multiplied by scalars to become $\lambda A = (\lambda a_{i,j})_{m,n}$.

Thus matrices of size m by n also have a linear algebraic structure.

Both commonly sized vector and matrices are closed under addition and scalar multiplication.

Our final tool is the dot product of n-vectors $x = (x_i)_n$ and $y = (y_i)_n$.

The dot product of two same n-dimensional vectors x and y is the scalar $x \cdot y = \sum_{i=1}^{n} x_i y_i$.

This may be the end of the very first hour of Linear Algebra class.

Now is the time to play with vectors and matrices. Form matrices by stacking equal dimensioned row vectors horizontally or equal dimensioned column vectors vertically.
It may also be advisable to introduce the class to Matlab or Python or other matrix capable software now and learn how to set up vectors and matrices inside the chosen software.
How to create a random entry matrix $A_{4,4}$, and then the block matrix $B_{8,8}$ that contains $7 \cdot A$ in its $(1,1)$ block, $-A$ in its $(1,2)$ block, $-2 \cdot A + 5 \cdot eye(n)$ in its $(2,1)$ block for the identity matrix $eye(4)$, as well as the zero matrix $zeros(4,4)$ in its $(2,2)$ diagonal block.
Maybe set up a computer help session or two with the TAs or yourself some afternoon in a campus computer lab and let students and you have fun and games ...

For simplicity, we will assume from now on and for a while that our vector spaces and matrices are all formed over the reals.

[Complex numbers only occur in the second half of the course when we study eigenvalues and eigenvectors of matrices.]

Any function f that maps $\mathbb{R}^n$ into $\mathbb{R}^m$ is comprised of m individual component functions f_j that each map $\mathbb{R}^n$ into $\mathbb{R}$.

Students should contemplate this fact for a while until comfortable with this component decomposition.

We will now study linear functions $f : \mathbb{R}^n \to \mathbb{R}^m$ more closely.

Indeed, we want to understand the following fact and prove it.

(1) *If $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is linear, then each of its m component functions f_i is linear.*

Proof :

We have $f(\alpha x + \beta y) = \begin{pmatrix} \vdots \\ f_j(\alpha x + \beta y) \\ \vdots \end{pmatrix}$ and

$$\alpha f(x) + \beta f(y) = \alpha \begin{pmatrix} \vdots \\ f_j(x) \\ \vdots \end{pmatrix} + \beta \begin{pmatrix} \vdots \\ f_j(y) \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \vdots \\ \alpha f_j(x) + \beta f_j(y) \\ \vdots \end{pmatrix},$$

when expressed for the ith component function f_j of f.

Assuming that f is linear, both left hand sides are equal.

And therefore the top and bottom right hand sides are also equal so that any component function f_j of f must be linear, too.

Now for the converse.

(2) *If each component function $f_j : \mathbb{R}^n \mapsto \mathbb{R}$ of a function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is linear, then f is linear.*

Proof:

$$f(\alpha x + \beta y) = \begin{pmatrix} \vdots \\ f_j(\alpha x + \beta y) \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \alpha f_j(x) + \beta f_j(y) \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \vdots \\ \alpha f_j(x) \\ \vdots \end{pmatrix} + \begin{pmatrix} \vdots \\ \beta f_j(y) \\ \vdots \end{pmatrix}$$

$$= \alpha \begin{pmatrix} \vdots \\ f_j(x) \\ \vdots \end{pmatrix} + \beta \begin{pmatrix} \vdots \\ f_j(y) \\ \vdots \end{pmatrix} = \alpha f(x) + \beta f(y)$$

for each $1 \leq j \leq m$ and f is linear.

Compatible n-vector sums of the kind $\alpha u + \beta v + \gamma w + ...$ are called linear combinations of the vectors $u, v, w, ...$ with scalars $\alpha, \beta, \gamma, ...$.

Each n-vector $x = (x_i)_n$ can be written as the linear combination of the standard unit vectors $e_i \in \mathbb{R}^n$.

Here each standard unit vectors e_i has $n - 1$ zero entries and a single entry of 1 in position $1 \leq i \leq n$.

Thus $x = (x_i)_n = x_1 e_1 + ... + x_n e_n$.

And due to the linearity of component functions $f_j : \mathbb{R}^n \to \mathbb{R}$ of linear functions f we have

$$f_j(x) = x_1 f_j(e_1) + ... + x_n f_j(e_n) = x \cdot (f_j(e_1), ..., f_j(e_n)) \in \mathbb{R}.$$

Hence all linear component functions f_j of a linear transformation f act as the dot product of the variable vector x and the constant vector $(f_j(e_1), ..., f_j(e_n))$ that defines $f_j : \mathbb{R}^n \to \mathbb{R}$.

This is Riesz's Representation Theorem in functional analysis for infinite dimensional spaces, developed here for finite dimensions n.

In finite dimensions it helps us to understand matrix multiplications and matrix $\times$ vector mappings.

How do we incorporate Riesz's Theorem into concrete matrix theory and matrix operations?

We stack the defining vectors $(f_i(e_1), ..., f_i(e_n))_n$ of each component function f_i of a linear function $f : \mathbb{R}^m \to \mathbb{R}^n$ in an m, n matrix

$$A = \begin{pmatrix} f_1(e_1) & \dots & f_1(e_n) \\ \vdots & & \vdots \\ f_m(e_1) & \dots & f_m(e_n) \end{pmatrix} = \begin{pmatrix} \vdots & \dots & \vdots \\ f(e_1) & \dots & f(e_n) \\ \vdots & \dots & \vdots \end{pmatrix}_{m,n} .$$

$A_{m,n}$ is called the standard matrix representation of the linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$.

$$A = \begin{pmatrix} f_1(e_1) & \dots & f_1(e_n) \\ \vdots & & \vdots \\ f_m(e_1) & \dots & f_m(e_n) \end{pmatrix} = \begin{pmatrix} \vdots & \dots & \vdots \\ f(e_1) & \dots & f(e_n) \\ \vdots & \dots & \vdots \end{pmatrix}_{m,n} .$$

The standard matrix representation A contains the images $\begin{pmatrix} \vdots \\ f(e_i) \\ \vdots \end{pmatrix}$ of the standard unit vectors $e_i \in \mathbb{R}^n$ under f in its n columns and - alternately - the defining vectors $\begin{pmatrix} f_j(e_1) & \cdots & f_j(e_n) \end{pmatrix}$ of each component function f_j of f in its m rows.

Recall: 'rows before columns'.

How to mate the standard matrix representations $A_{m,n}$ of a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ with a vector $x \in \mathbb{R}^n$ to evaluate $f(x) = A \cdot x$?

To evaluate the necessary dot products of $(f_j(e_1), ..., f_j(e_n))_n$ and $x \in \mathbb{R}^n$ by using $A_{m,n}$, we write x as a column vector, place it to A's right and evaluate Ax $(= f(x))$ as

$$Ax = \begin{pmatrix} f_1(e_1) & \dots & f_1(e_n) \\ \vdots & & \vdots \\ f_m(e_1) & \dots & f_m(e_n) \end{pmatrix}_{m,n} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_n$$

$$= \begin{pmatrix} \vdots \\ \begin{pmatrix} f_j(e_1) & \dots & f_j(e_n) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ \vdots \end{pmatrix}_m$$

as m repeated matrix row $\times$ column vector dot products.

Thus we have achieved a concrete way to express general linear transformations between finite dimensional vector spaces by using matrix $\times$ vector products.

The Linear Transformation Theorem

Every linear transformation $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ ***can be expressed as a constant matrix*** $A \times$ ***vector product*** $A_{m,n}\, x_n$***.***

This establishes the equivalence of classical finite dimensional Linear Algebra and Matrix Theory.

Matrix Theory is concrete and codeable. It can answer and solve all questions of abstract Linear Algebra.

Matrix Theory is modern and the language for computing.

It, rather than abstract reasoning, is used in all applications nowadays. And that we shall teach.

This ends the introductory first lesson plan that covers 2 or 3 class hours of fundamentals in Linear Algebra and in the concrete matrix/vector setting.

Some ideas for homework (and classwork on the next class day):
Think of a matrix $A_{m,n}$ as a stack of n column vectors as one would stack books $|$ on a shelf, left to right $|\,|\,|$ ....

What does the matrix $\times$ vector product $A \cdot x$ do with the columns $\begin{pmatrix} \vdots \\ c_i \\ \vdots \end{pmatrix}$ *in A?*

$$A_{m,n} \cdot x = \begin{pmatrix} \vdots & \dots & \vdots \\ c_1 & & c_n \\ \vdots & \dots & \vdots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} \vdots \\ c_1 \\ \vdots \end{pmatrix} + x_2 \begin{pmatrix} \vdots \\ c_2 \\ \vdots \end{pmatrix} + \dots$$

$$\dots + x_n \begin{pmatrix} \vdots \\ c_n \\ \vdots \end{pmatrix}.$$

Hint: multiplying $A_{m,n}$ by an n-vector x on the right creates a linear combination of the column vectors in A.

How would the students multiply an m by n matrix $A_{m,n}$ by an m-vector y from the left?

$$[\ \textit{What should}\ \ y \cdot A_{m,n} = \begin{pmatrix} y_1 & \cdots & y_m \end{pmatrix} \begin{pmatrix} \cdots & r_1 & \cdots \\ & \vdots & \\ \cdots & r_m & \cdots \end{pmatrix} \textit{mean?}\ ]$$

A row stack ...

Ask students to create matrices and vectors in either column or row form and try to multiply them in class. Some products will work and others will fail.

Why, and how is that recognized?

Our first lesson on linear transformations and matrix times vector multiplications might take two to four actual class hours and this leads us simple but tedious applications in the second lesson.

The second lesson plan will again contain outlines for 2 to 4 class sessions.

Sometimes full of concrete number crunching, sometimes dealing with abstractions and a few small proofs, too.

The subjects of Lesson Plan 2 are :

row reduction of rectangular matrices to row echelon form,
interpreting row reduced forms in terms of spanning sets of matrix columns or rows,
minimal spanning sets or bases for vector subspaces, and
to learn how to solve solve systems of linear equations via row reduction.

The second lesson plan will take a class to near the halfway point of a modern first Linear Algebra and Matrix Theory course.