ILAS-ED Seminar

Zoom ID 929 2504 4372 Meeting, September 27, 2022

(Matrix) Zeros of polynomials

Damjan Kobal, University of Ljubljana

Mathematics learning

Mathematics learning

Mathematics learning

from intuitive observations and intuitive insights

Mathematics learning

from intuitive observations and intuitive insights

to abstract notions and understanding

Mathematics learning

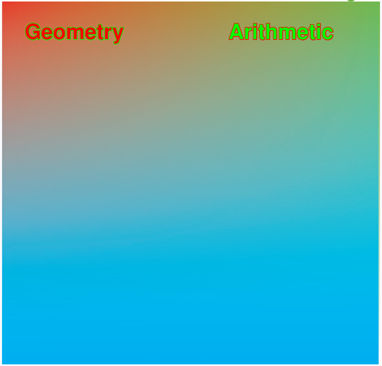
from intuitive observations and intuitive insights

Geometry

to abstract notions and understanding

Mathematics learning

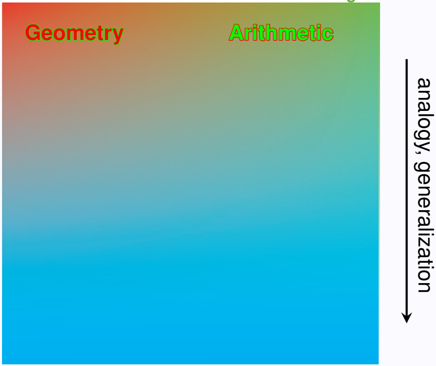
from intuitive observations and intuitive insights



to abstract notions and understanding

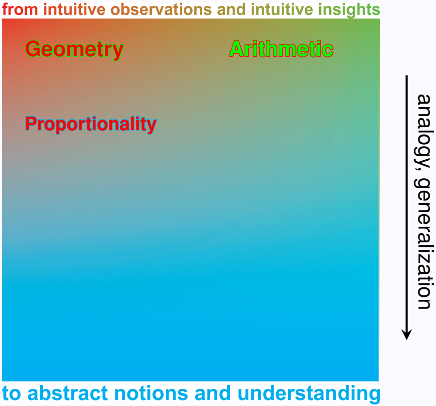
Mathematics learning

from intuitive observations and intuitive insights



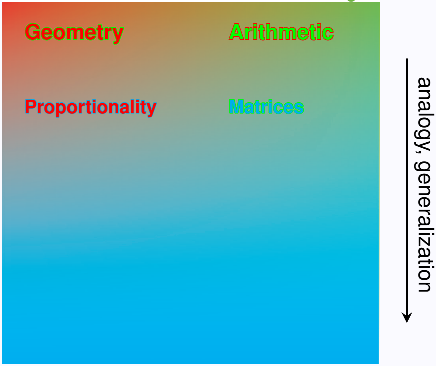
to abstract notions and understanding

Mathematics learning



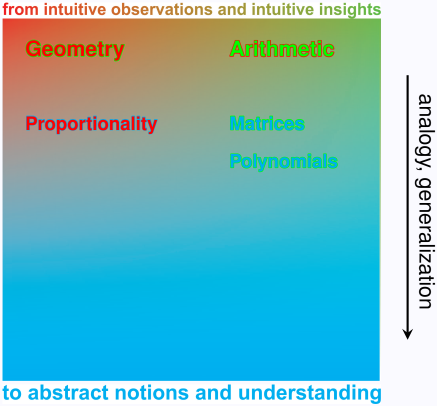
Mathematics learning

from intuitive observations and intuitive insights

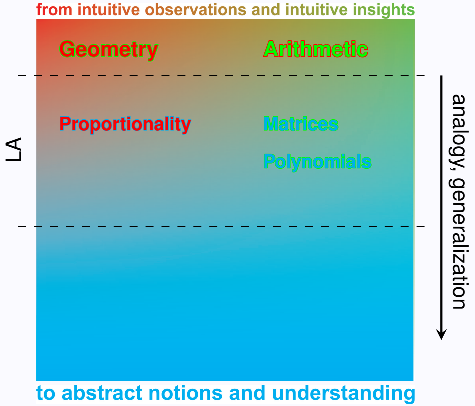


to abstract notions and understanding

Mathematics learning



Mathematics learning



... from intuition to abstraction geometry-algebra sample ...

https://www.geogebra.org/m/vguhhxzj

Matrix zeros of polynomials

Matrix zeros of polynomials

D. Kobal, *Matrix zeros of polynomials*, The Mathematical Gazette, Cambridge Univeristy Press, Vol. 104, March 2020, p. 27 - 35.

Matrix zeros of polynomials

D. Kobal, *Matrix zeros of polynomials*, The Mathematical Gazette, Cambridge Univeristy Press, Vol. 104, March 2020, p. 27 - 35.

Presented also at the *Linear algebra education minisymposia*, ILAS 2022, Galway, Ireland, June 2022.

Introduction

Introduction

- The concepts of polynomials and matrices generalize the elementary arithmetic of numbers.

Introduction

- Our elementary exploration of polynomials and matrices offers an interesting matrix analogue to the concept of a zero of a polynomial.

Introduction

- The discussion offers an opportunity for better comprehension of the fundamental concepts of polynomials and matrices.

Introduction

- As an application we are lead to the meaningful questions of linear algebra and to an easy proof of otherwise advanced Cayley - Hamilton theorem.

The arithmetic of matrices and polynomials

The arithmetic of matrices and polynomials

- After mastering the basic polynomial and matrix calculations, including the concepts of square matrices (over a field, usually $\mathbb{C}$), identity and inverse matrices, we can discuss also ‘polynomials of matrices’ and ‘matrices of polynomials.’

The arithmetic of matrices and polynomials

- After mastering the basic polynomial and matrix calculations, including the concepts of square matrices (over a field, usually $\mathbb{C}$), identity and inverse matrices, we can discuss also 'polynomials of matrices' and 'matrices of polynomials.'
- In what follows, all our matrices will be square matrices.

The arithmetic of matrices and polynomials

- After mastering the basic polynomial and matrix calculations, including the concepts of square matrices (over a field, usually $\mathbb{C}$), identity and inverse matrices, we can discuss also 'polynomials of matrices' and 'matrices of polynomials.'
- In what follows, all our matrices will be square matrices.

Polynomials of matrices

The arithmetic of matrices and polynomials

- After mastering the basic polynomial and matrix calculations, including the concepts of square matrices (over a field, usually $\mathbb{C}$), identity and inverse matrices, we can discuss also 'polynomials of matrices' and 'matrices of polynomials.'
- In what follows, all our matrices will be square matrices.

Polynomials of matrices

- $p(x), M \longrightarrow p(M)$

The arithmetic of matrices and polynomials

- After mastering the basic polynomial and matrix calculations, including the concepts of square matrices (over a field, usually $\mathbb{C}$), identity and inverse matrices, we can discuss also 'polynomials of matrices' and 'matrices of polynomials.'
- In what follows, all our matrices will be square matrices.

Polynomials of matrices

- $p(x), M \longrightarrow p(M)$
- $p(x) = x^2 - 3x + 2, M = \begin{bmatrix} 3 & 2 & -2 \\ 1 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix}$

The arithmetic of matrices and polynomials

- After mastering the basic polynomial and matrix calculations, including the concepts of square matrices (over a field, usually $\mathbb{C}$), identity and inverse matrices, we can discuss also 'polynomials of matrices' and 'matrices of polynomials.'
- In what follows, all our matrices will be square matrices.

Polynomials of matrices

- $p(x), M \longrightarrow p(M)$
- $p(x) = x^2 - 3x + 2, M = \begin{bmatrix} 3 & 2 & -2 \\ 1 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix} \longrightarrow p(M) = \begin{bmatrix} 0 & -2 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix}.$

Matrices of polynomials

Matrices of polynomials

- For example

$$\begin{bmatrix} x^2 + x + 1 & x \\ 1 & 2x^2 \end{bmatrix} =$$

Matrices of polynomials

- For example

$$\begin{bmatrix} x^2 + x + 1 & x \\ 1 & 2x^2 \end{bmatrix} = x^2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + x \cdot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}. \tag{1}$$

Matrices of polynomials

- For example

$$\begin{bmatrix} x^2+x+1 & x \\ 1 & 2x^2 \end{bmatrix} = x^2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + x \cdot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}. \qquad (1)$$

- Or

$$\begin{bmatrix} x^2+x+1 & x \\ 1 & 2x^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot x^2 + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot x + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Matrices of polynomials

- For example

$$\begin{bmatrix} x^2+x+1 & x \\ 1 & 2x^2 \end{bmatrix} = x^2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + x \cdot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}. \tag{1}$$

- Or

$$\begin{bmatrix} x^2+x+1 & x \\ 1 & 2x^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot x^2 + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot x + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

- Notation (*polynomial matrix*) $M(x) = \begin{bmatrix} x^2+x+1 & x \\ 1 & 2x^2 \end{bmatrix}$

Matrices of polynomials

- For example

$$\begin{bmatrix} x^2+x+1 & x \\ 1 & 2x^2 \end{bmatrix} = x^2 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + x \cdot \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}. \qquad (1)$$

- Or

$$\begin{bmatrix} x^2+x+1 & x \\ 1 & 2x^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot x^2 + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \cdot x + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

- Notation (*polynomial matrix*) $M(x) = \begin{bmatrix} x^2+x+1 & x \\ 1 & 2x^2 \end{bmatrix}$

Remark 1

For a polynomial matrix $M(x)$ and matrix A, it makes no sense to talk about $M(A)$.

Matrices and zeros of polynomials

Matrices and zeros of polynomials

- A number m is a zero of the polynomial $p(x)$ if and only if $p(m) = 0$.

Matrices and zeros of polynomials

- A number m is a zero of the polynomial $p(x)$ if and only if $p(m) = 0$.
- For matrix M and polynomial $p(x)$, we might get $p(M) = 0$

Matrices and zeros of polynomials

- A number m is a zero of the polynomial $p(x)$ if and only if $p(m) = 0$.
- For matrix M and polynomial $p(x)$, we might get $p(M) = 0$
- For example, for $p(x) = x^2 - 3x + 2$ for

$$M = \begin{bmatrix} 3 & 2 & -2 \\ 1 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix} \quad \text{and for} \quad N = \begin{bmatrix} 3 & 2 & -2 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}$$

Matrices and zeros of polynomials

- A number m is a zero of the polynomial $p(x)$ if and only if $p(m) = 0$.
- For matrix M and polynomial $p(x)$, we might get $p(M) = 0$
- For example, for $p(x) = x^2 - 3x + 2$ for

$$M = \begin{bmatrix} 3 & 2 & -2 \\ 1 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix} \text{ and for } N = \begin{bmatrix} 3 & 2 & -2 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}$$

we get

$$p(M) = \begin{bmatrix} 0 & -2 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix} \text{ and } p(N) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Matrices and zeros of polynomials

- A number m is a zero of the polynomial $p(x)$ if and only if $p(m) = 0$.
- For matrix M and polynomial $p(x)$, we might get $p(M) = 0$
- For example, for $p(x) = x^2 - 3x + 2$ for

$$M = \begin{bmatrix} 3 & 2 & -2 \\ 1 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix} \text{ and for } N = \begin{bmatrix} 3 & 2 & -2 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}$$

  we get

$$p(M) = \begin{bmatrix} 0 & -2 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix} \text{ and } p(N) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- *Definition :* We say that a matrix M is a *matrix zero* of the polynomial $p(x)$ if and only if $p(M)$ is the zero matrix.

Matrices and zeros of polynomials

- A number m is a zero of the polynomial $p(x)$ if and only if $p(m) = 0$.
- For matrix M and polynomial $p(x)$, we might get $p(M) = 0$
- For example, for $p(x) = x^2 - 3x + 2$ for

$$M = \begin{bmatrix} 3 & 2 & -2 \\ 1 & 1 & -1 \\ 2 & 2 & -1 \end{bmatrix} \text{ and for } N = \begin{bmatrix} 3 & 2 & -2 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}$$

  we get

$$p(M) = \begin{bmatrix} 0 & -2 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix} \text{ and } p(N) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- *Definition :* We say that a matrix M is a *matrix zero* of the polynomial $p(x)$ if and only if $p(M)$ is the zero matrix.
- Could anything be said about matrix zeros of a given polynomial?

Matrix zero of a polynomial - exploration

Matrix zero of a polynomial - exploration

- Could we find a matrix zero of a simple polynomial

$$p(x) = x^2 - 3x + 2 = (x-1)(x-2)?$$

Matrix zero of a polynomial - exploration

- Could we find a matrix zero of a simple polynomial

$$p(x) = x^2 - 3x + 2 = (x-1)(x-2)?$$

One such matrix zero, which looks quite arbitrary, was given above.

Matrix zero of a polynomial - exploration

- Could we find a matrix zero of a simple polynomial

$$p(x) = x^2 - 3x + 2 = (x-1)(x-2)?$$

One such matrix zero, which looks quite arbitrary, was given above. Could we find another? Or maybe many others?

Matrix zero of a polynomial - exploration

- Could we find a matrix zero of a simple polynomial

$$p(x) = x^2 - 3x + 2 = (x - 1)(x - 2)?$$

  One such matrix zero, which looks quite arbitrary, was given above. Could we find another? Or maybe many others?

- Zeros 1 and 2 as 'one-dimensional' matrices $[1]$ and $[2]$.

Matrix zero of a polynomial - exploration

- Could we find a matrix zero of a simple polynomial

$$p(x) = x^2 - 3x + 2 = (x-1)(x-2)?$$

  One such matrix zero, which looks quite arbitrary, was given above. Could we find another? Or maybe many others?
- Zeros 1 and 2 as 'one-dimensional' matrices $[1]$ and $[2]$. Zeros of the form $1 \cdot I$ or $2 \cdot I$.

Matrix zero of a polynomial - exploration

- Could we find a matrix zero of a simple polynomial

$$p(x) = x^2 - 3x + 2 = (x-1)(x-2)?$$

  One such matrix zero, which looks quite arbitrary, was given above. Could we find another? Or maybe many others?

- Zeros 1 and 2 as 'one-dimensional' matrices $[1]$ and $[2]$. Zeros of the form $1 \cdot I$ or $2 \cdot I$.

- Diagonal matrices D

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

Matrix zero of a polynomial - exploration

- Could we find a matrix zero of a simple polynomial

$$p(x) = x^2 - 3x + 2 = (x-1)(x-2)?$$

  One such matrix zero, which looks quite arbitrary, was given above. Could we find another? Or maybe many others?

- Zeros 1 and 2 as 'one-dimensional' matrices $[1]$ and $[2]$. Zeros of the form $1 \cdot I$ or $2 \cdot I$.
- Diagonal matrices D

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

Matrix zero of a polynomial - exploration

- Could we find a matrix zero of a simple polynomial

$$p(x) = x^2 - 3x + 2 = (x-1)(x-2)?$$

  One such matrix zero, which looks quite arbitrary, was given above. Could we find another? Or maybe many others?

- Zeros 1 and 2 as 'one-dimensional' matrices $[1]$ and $[2]$. Zeros of the form $1 \cdot I$ or $2 \cdot I$.
- Diagonal matrices D

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Matrix zero of a polynomial - exploration

- Could we find a matrix zero of a simple polynomial

$$p(x) = x^2 - 3x + 2 = (x-1)(x-2)?$$

  One such matrix zero, which looks quite arbitrary, was given above. Could we find another? Or maybe many others?

- Zeros 1 and 2 as 'one-dimensional' matrices $[1]$ and $[2]$. Zeros of the form $1 \cdot I$ or $2 \cdot I$.
- Diagonal matrices D

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix zero of a polynomial - exploration

- Could we find a matrix zero of a simple polynomial

$$p(x) = x^2 - 3x + 2 = (x-1)(x-2)?$$

One such matrix zero, which looks quite arbitrary, was given above. Could we find another? Or maybe many others?

- Zeros 1 and 2 as 'one-dimensional' matrices $[1]$ and $[2]$. Zeros of the form $1 \cdot I$ or $2 \cdot I$.
- Diagonal matrices D

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

seem mysterious, but anticipated, 'matrix zeros.'

Matrix zero of a polynomial - exploration

- Could we find a matrix zero of a simple polynomial

$$p(x) = x^2 - 3x + 2 = (x-1)(x-2)?$$

  One such matrix zero, which looks quite arbitrary, was given above. Could we find another? Or maybe many others?

- Zeros 1 and 2 as 'one-dimensional' matrices $[1]$ and $[2]$. Zeros of the form $1 \cdot I$ or $2 \cdot I$.
- Diagonal matrices D

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  seem mysterious, but anticipated, 'matrix zeros.'

- Calculating $(D-I)(D-2I)$

Matrix zero of a polynomial - exploration

- Could we find a matrix zero of a simple polynomial

$$p(x) = x^2 - 3x + 2 = (x-1)(x-2)?$$

  One such matrix zero, which looks quite arbitrary, was given above. Could we find another? Or maybe many others?

- Zeros 1 and 2 as 'one-dimensional' matrices $[1]$ and $[2]$. Zeros of the form $1 \cdot I$ or $2 \cdot I$.
- Diagonal matrices D

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  seem mysterious, but anticipated, 'matrix zeros.'

- Calculating $(D - I)(D - 2I)$ gives a nice intuitive understanding of the 'block matrix multiplication.

Matrix zero of a polynomial - exploration

- Could we find a matrix zero of a simple polynomial

$$p(x) = x^2 - 3x + 2 = (x-1)(x-2)?$$

  One such matrix zero, which looks quite arbitrary, was given above. Could we find another? Or maybe many others?

- Zeros 1 and 2 as 'one-dimensional' matrices $[1]$ and $[2]$. Zeros of the form $1 \cdot I$ or $2 \cdot I$.
- Diagonal matrices D

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  seem mysterious, but anticipated, 'matrix zeros.'

- Calculating $(D - I)(D - 2I)$ gives a nice intuitive understanding of the 'block matrix multiplication. (These observations will be valuable later, when we learn about eigenvalues.)

Matrix zero of a polynomial - further exploration

Matrix zero of a polynomial - further exploration

- $$\left.\begin{array}{r} p(D) = 0 \\ M = P \cdot D \cdot P^{-1} \end{array}\right\} \longrightarrow p(M) = 0$$

Matrix zero of a polynomial - further exploration

- $$\left.\begin{array}{r} p(D) = 0 \\ M = P \cdot D \cdot P^{-1} \end{array}\right\} \longrightarrow p(M) = 0$$

- At this point students should ... notice that

Matrix zero of a polynomial - further exploration

- $$\left.\begin{array}{r} p(D) = 0 \\ M = P \cdot D \cdot P^{-1} \end{array}\right\} \longrightarrow p(M) = 0$$

- At this point students should ... notice that
  - ‘matrix zeros’ of a polynomial are quite complex,

Matrix zero of a polynomial - further exploration

- $$\left.\begin{array}{r} p(D) = 0 \\ M = P \cdot D \cdot P^{-1} \end{array}\right\} \longrightarrow p(M) = 0$$
- At this point students should ... notice that
  - ‘matrix zeros’ of a polynomial are quite complex,
  - already a (simple) polynomial might have infinitely many ‘matrix zeros,’

Matrix zero of a polynomial - further exploration

- $$\left.\begin{aligned} p(D) &= 0 \\ M = P \cdot D \cdot P^{-1} \end{aligned}\right\} \longrightarrow p(M) = 0$$

- At this point students should ... notice that
  - ‘matrix zeros’ of a polynomial are quite complex,
  - already a (simple) polynomial might have infinitely many ‘matrix zeros,’
  - considering *D* and *P* as above, we have a pattern to find many ‘matrix zeros,’

Matrix zero of a polynomial - further exploration

- $$\left.\begin{aligned} p(D) &= 0 \\ M &= P \cdot D \cdot P^{-1} \end{aligned}\right\} \longrightarrow p(M) = 0$$
- At this point students should ... notice that
  - 'matrix zeros' of a polynomial are quite complex,
  - already a (simple) polynomial might have infinitely many 'matrix zeros,'
  - considering D and P as above, we have a pattern to find many 'matrix zeros,'
  - this pattern is well-hidden within obtained 'matrix zeros' (by free choice of the invertible matrix P, the obtained matrix M looks quite arbitrary).

Theorem 1

The number m is a zero of the polynomial $p(x)$ if and only if $p(x) = r(x) \cdot (x - m)$ for some polynomial $r(x)$.

Theorem 1

The number m is a zero of the polynomial $p(x)$ if and only if $p(x) = r(x) \cdot (x - m)$ for some polynomial $r(x)$.

Proof.

$(\Longleftarrow)$

Theorem 1

The number m is a zero of the polynomial $p(x)$ if and only if $p(x) = r(x) \cdot (x - m)$ for some polynomial $r(x)$.

Proof.

$(\Longleftarrow)$ $p(x) = r(x) \cdot (x - m)$

Theorem 1

The number m is a zero of the polynomial $p(x)$ if and only if $p(x) = r(x) \cdot (x - m)$ for some polynomial $r(x)$.

Proof.

$(\Longleftarrow)$ $p(x) = r(x) \cdot (x - m) \longrightarrow p(m) = r(x) \cdot (m - m) = 0.$

Theorem 1

The number m is a zero of the polynomial $p(x)$ if and only if $p(x) = r(x) \cdot (x - m)$ for some polynomial $r(x)$.

Proof.

$(\Longleftarrow)$ $p(x) = r(x) \cdot (x - m) \longrightarrow p(m) = r(x) \cdot (m - m) = 0.$

$(\Longrightarrow)$

Theorem 1

The number m is a zero of the polynomial $p(x)$ if and only if $p(x) = r(x) \cdot (x - m)$ for some polynomial $r(x)$.

Proof.

$(\Longleftarrow)$ $p(x) = r(x) \cdot (x - m) \longrightarrow p(m) = r(x) \cdot (m - m) = 0$.

$(\Longrightarrow)$ $p(m) = 0$

Theorem 1

The number m is a zero of the polynomial $p(x)$ if and only if $p(x) = r(x) \cdot (x - m)$ for some polynomial $r(x)$.

Proof.

$(\Longleftarrow)$ $p(x) = r(x) \cdot (x - m) \longrightarrow p(m) = r(x) \cdot (m - m) = 0.$

$(\Longrightarrow)$ $p(m) = 0$

$$x^i - m^i = \quad (x^{i-1} + x^{i-2}m + \cdots + xm^{i-2} + m^{i-1})(x - m)$$

Theorem 1

The number m is a zero of the polynomial $p(x)$ if and only if $p(x) = r(x) \cdot (x - m)$ for some polynomial $r(x)$.

Proof.

$(\Longleftarrow)$ $p(x) = r(x) \cdot (x - m) \longrightarrow p(m) = r(x) \cdot (m - m) = 0.$

$(\Longrightarrow)$ $p(m) = 0$

$$\begin{aligned} x^i - m^i = &\quad (x^{i-1} + x^{i-2}m + \cdots + xm^{i-2} + m^{i-1})(x - m) \\ p(x) - p(m) = &\quad \sum_{i=1}^{n} a_i \cdot (x^{i-1} + x^{i-2}m + \cdots + xm^{i-2} + m^{i-1})(x - m) \end{aligned}$$

Theorem 1

The number m is a zero of the polynomial $p(x)$ *if and only if* $p(x) = r(x) \cdot (x - m)$ *for some polynomial* $r(x)$.

Proof.

$(\Longleftarrow)$ $p(x) = r(x) \cdot (x - m) \longrightarrow p(m) = r(x) \cdot (m - m) = 0$.

$(\Longrightarrow)$ $p(m) = 0$

$$\begin{aligned} x^i - m^i &= (x^{i-1} + x^{i-2}m + \cdots + xm^{i-2} + m^{i-1})(x - m) \\ p(x) - p(m) &= \sum_{i=1}^{n} a_i \cdot (x^{i-1} + x^{i-2}m + \cdots + xm^{i-2} + m^{i-1})(x - m) \\ &= r(x) \cdot (x - m) \end{aligned}$$

Theorem 1

The number m is a zero of the polynomial $p(x)$ if and only if $p(x) = r(x) \cdot (x - m)$ for some polynomial $r(x)$.

Proof.

$(\Longleftarrow)$ $p(x) = r(x) \cdot (x - m) \longrightarrow p(m) = r(x) \cdot (m - m) = 0.$

$(\Longrightarrow)$ $p(m) = 0$

$$\begin{aligned} x^i - m^i &= (x^{i-1} + x^{i-2}m + \cdots + xm^{i-2} + m^{i-1})(x - m) \\ p(x) - p(m) &= \sum_{i=1}^{n} a_i \cdot (x^{i-1} + x^{i-2}m + \cdots + xm^{i-2} + m^{i-1})(x - m) \\ &= r(x) \cdot (x - m) \\ p(x) &= p(m) + r(x) \cdot (x - m) = r(x) \cdot (x - m) \end{aligned}$$

□

What could be a sensible analogue of Theorem 1 for matrices?

$$p(x) = r(x) \cdot (x - m)$$

What could be a sensible analogue of Theorem 1 for matrices?

$$p(x) = r(x) \cdot (x - m)$$

$$(X - M)$$

What could be a sensible analogue of Theorem 1 for matrices?

$$p(x) = r(x) \cdot (x - m)$$

$$R(x) \cdot (X - M)$$

What could be a sensible analogue of Theorem 1 for matrices?

$$p(x) = r(x) \cdot (x - m)$$

$$R(x) \cdot (x \cdot I - M)$$

What could be a sensible analogue of Theorem 1 for matrices?

$$p(x) = r(x) \cdot (x - m)$$

$$p(x) \cdot I = R(x) \cdot (x \cdot I - M)$$

Theorem 2

The matrix M is a matrix zero of the polynomial $p(x)$ if and only if

$$p(x) \cdot I = R(x) \cdot (xI - M)$$

for some polynomial matrix $R(x)$, where I is the identity matrix.

Theorem 2

The matrix M is a matrix zero of the polynomial $p(x)$ if and only if

$$p(x) \cdot I = R(x) \cdot (xI - M)$$

for some polynomial matrix $R(x)$, where I is the identity matrix.

Example

Theorem 2

The matrix M is a matrix zero of the polynomial $p(x)$ if and only if

$$p(x) \cdot I = R(x) \cdot (xI - M)$$

for some polynomial matrix $R(x)$, where I is the identity matrix.

Example

For $p(x) = x^2 - 3x + 2$ and $N = \begin{bmatrix} 3 & 2 & -2 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}$, we have $p(N) = 0$.

Theorem 2

The matrix M is a matrix zero of the polynomial $p(x)$ if and only if

$$p(x) \cdot I = R(x) \cdot (xI - M)$$

for some polynomial matrix $R(x)$, where I is the identity matrix.

Example

For $p(x) = x^2 - 3x + 2$ and $N = \begin{bmatrix} 3 & 2 & -2 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}$, we have $p(N) = 0$.

$$(x^2 - 3x + 2) \cdot I = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \cdot (xI - N)$$

Theorem 2

The matrix M is a matrix zero of the polynomial $p(x)$ if and only if

$$p(x) \cdot I = R(x) \cdot (xI - M)$$

for some polynomial matrix $R(x)$, where I is the identity matrix.

Example

For $p(x) = x^2 - 3x + 2$ and $N = \begin{bmatrix} 3 & 2 & -2 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}$, we have $p(N) = 0$.

$$(x^2 - 3x + 2) \cdot I = \begin{bmatrix} x & 2 & -2 \\ 1 & x-1 & -1 \\ 2 & 2 & x-4 \end{bmatrix} \cdot (xI - N)$$

Remark 2

As in Theorem 1

$$p(M) = 0 \longrightarrow p(x) \cdot I = R(x) \cdot (xI - M)$$

Remark 2

As in Theorem 1

$$p(M) = 0 \longrightarrow p(x) \cdot I = R(x) \cdot (xI - M)$$

Conversely (trivial in ‘number-case’)

Remark 2

As in Theorem 1

$$p(M) = 0 \longrightarrow p(x) \cdot I = R(x) \cdot (xI - M)$$

Conversely (trivial in 'number-case')

$$p(x) \cdot I = R(x) \cdot (xI - M) \not\longrightarrow p(M) = 0$$

Remark 2

As in Theorem 1

$$p(M) = 0 \longrightarrow p(x) \cdot I = R(x) \cdot (xI - M)$$

Conversely (trivial in 'number-case')

$$p(x) \cdot I = R(x) \cdot (xI - M) \not\longrightarrow p(M) = 0$$

We can not simply calculate $p(M)I = R(M) \cdot (M \cdot I - M) = 0$

Remark 2

As in Theorem 1

$$p(M) = 0 \longrightarrow p(x) \cdot I = R(x) \cdot (xI - M)$$

Conversely (trivial in 'number-case')

$$p(x) \cdot I = R(x) \cdot (xI - M) \not\longrightarrow p(M) = 0$$

We can not simply calculate $p(M)I = R(M) \cdot (M \cdot I - M) = 0$*, as* $R(M)$ *does not make sense.*

Remark 2

As in Theorem 1

$$p(M) = 0 \longrightarrow p(x) \cdot I = R(x) \cdot (xI - M)$$

Conversely (trivial in 'number-case')

$$p(x) \cdot I = R(x) \cdot (xI - M) \not\longrightarrow p(M) = 0$$

We can not simply calculate $p(M)I = R(M) \cdot (M \cdot I - M) = 0$*, as* $R(M)$ *does not make sense.*

Remark 3

Somehow similar mistake:

Remark 2

As in Theorem 1

$$p(M) = 0 \longrightarrow p(x) \cdot I = R(x) \cdot (xI - M)$$

Conversely (trivial in 'number-case')

$$p(x) \cdot I = R(x) \cdot (xI - M) \not\longrightarrow p(M) = 0$$

We can not simply calculate $p(M)I = R(M) \cdot (M \cdot I - M) = 0$*, as* $R(M)$ *does not make sense.*

Remark 3

Somehow similar mistake: For characteristic polynomial

$$\chi_M(x) = \det(xI - M) \not\longrightarrow \chi_M(M) = \det(M \cdot I - M) = 0$$

Remark 2

As in Theorem 1

$$p(M) = 0 \longrightarrow p(x) \cdot I = R(x) \cdot (xI - M)$$

Conversely (trivial in 'number-case')

$$p(x) \cdot I = R(x) \cdot (xI - M) \not\longrightarrow p(M) = 0$$

We can not simply calculate $p(M)I = R(M) \cdot (M \cdot I - M) = 0$*, as* $R(M)$ *does not make sense.*

Remark 3

Somehow similar mistake: For characteristic polynomial

$$\chi_M(x) = \det(xI - M) \not\longrightarrow \chi_M(M) = \det(M \cdot I - M) = 0$$

Full understanding of Cayley-Hamilton theorem is hard ...,

Remark 2

As in Theorem 1

$$p(M) = 0 \longrightarrow p(x) \cdot I = R(x) \cdot (xI - M)$$

Conversely (trivial in 'number-case')

$$p(x) \cdot I = R(x) \cdot (xI - M) \not\longrightarrow p(M) = 0$$

We can not simply calculate $p(M)I = R(M) \cdot (M \cdot I - M) = 0$, *as* $R(M)$ *does not make sense.*

Remark 3

Somehow similar mistake: For characteristic polynomial

$$\chi_M(x) = \det(xI - M) \not\longrightarrow \chi_M(M) = \det(M \cdot I - M) = 0$$

Full understanding of Cayley-Hamilton theorem is hard ..., but every student to whom we consider 'worthy' to introduce the 'characteristic polynomial', should understand, that in this 'equation' we have a '$n \times n$*' matrix on the left and a number '* 0 *' on the right.*

Analogy between ‘number’ and ‘matrix’ case.

Analogy between ‘number’ and ‘matrix’ case.

Lemma 1

If $p(x)$ is a polynomial such that $p(x) \cdot (x - a) = c$, where a and c are constants, then $c = 0$ and $p(x) \equiv 0$.

Analogy between ‘number’ and ‘matrix’ case.

Lemma 1

If $p(x)$ is a polynomial such that $p(x) \cdot (x - a) = c$, where a and c are constants, then $c = 0$ and $p(x) \equiv 0$.

Proof.

$$p(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

Analogy between 'number' and 'matrix' case.

Lemma 1

If $p(x)$ is a polynomial such that $p(x) \cdot (x - a) = c$, where a and c are constants, then $c = 0$ and $p(x) \equiv 0$.

Proof.

$$p(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

$$c = p(x) \cdot (x - a) = b_n x^{n+1} + (b_{n-1} - a b_n) x^n + \cdots + (b_0 - b_1 a) x - b_0 a.$$

Analogy between 'number' and 'matrix' case.

Lemma 1

If $p(x)$ is a polynomial such that $p(x) \cdot (x - a) = c$, where a and c are constants, then $c = 0$ and $p(x) \equiv 0$.

Proof.

$$p(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

$$c = p(x) \cdot (x - a) = b_n x^{n+1} + (b_{n-1} - ab_n) x^n + \cdots + (b_0 - b_1 a) x - b_0 a.$$

$$\longrightarrow b_n = 0$$

Analogy between ‘number’ and ‘matrix’ case.

Lemma 1

If $p(x)$ is a polynomial such that $p(x) \cdot (x - a) = c$, where a and c are constants, then $c = 0$ and $p(x) \equiv 0$.

Proof.

$$p(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

$$c = p(x) \cdot (x - a) = b_n x^{n+1} + (b_{n-1} - a b_n) x^n + \cdots + (b_0 - b_1 a) x - b_0 a.$$

$$\longrightarrow b_n = 0 \longrightarrow b_{n-1} = 0$$

Analogy between ‘number’ and ‘matrix’ case.

Lemma 1

If $p(x)$ is a polynomial such that $p(x) \cdot (x - a) = c$, where a and c are constants, then $c = 0$ and $p(x) \equiv 0$.

Proof.

$$p(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

$$c = p(x) \cdot (x - a) = b_n x^{n+1} + (b_{n-1} - ab_n) x^n + \cdots + (b_0 - b_1 a) x - b_0 a.$$

$$\longrightarrow b_n = 0 \longrightarrow b_{n-1} = 0 \longrightarrow \ldots \longrightarrow b_0 = 0$$

Analogy between 'number' and 'matrix' case.

Lemma 1

If $p(x)$ is a polynomial such that $p(x) \cdot (x - a) = c$, where a and c are constants, then $c = 0$ and $p(x) \equiv 0$.

Proof.

$$p(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$$

$$c = p(x) \cdot (x - a) = b_n x^{n+1} + (b_{n-1} - a b_n) x^n + \cdots + (b_0 - b_1 a) x - b_0 a.$$

$$\longrightarrow b_n = 0 \longrightarrow b_{n-1} = 0 \longrightarrow \ldots \longrightarrow b_0 = 0 \longrightarrow p(x) \equiv 0 \longrightarrow c = 0$$

□

Lemma 2

Lemma 2

If $N(x)$ *is a polynomial matrix such that* $N(x) \cdot (xI - A) = M$*, where* A *and* M *are matrices (with constant entries), then* M *must be the zero matrix and* $N(x) \equiv 0$*.*

Lemma 2

If $N(x)$ is a polynomial matrix such that $N(x) \cdot (xI - A) = M$, where A and M are matrices (with constant entries), then M must be the zero matrix and $N(x) \equiv 0$.

Proof.

Lemma 2

If $N(x)$ is a polynomial matrix such that $N(x) \cdot (xI - A) = M$, where A and M are matrices (with constant entries), then M must be the zero matrix and $N(x) \equiv 0$.

Proof.

$N(x)$ can be written (see (1))

$$N(x) = N_n x^n + N_{n-1} x^{n-1} + \cdots + N_1 x + N_0$$

Lemma 2

If $N(x)$ *is a polynomial matrix such that* $N(x) \cdot (xI - A) = M$*, where* A *and* M *are matrices (with constant entries), then* M *must be the zero matrix and* $N(x) \equiv 0$*.*

Proof.

$N(x)$ can be written (see (1))

$$N(x) = N_n x^n + N_{n-1} x^{n-1} + \cdots + N_1 x + N_0$$

$$\longrightarrow M = N_n x^{n+1} + (N_{n-1} - N_n \cdot A) x^n + \cdots + (N_0 - N_1 \cdot A) x - N_0 \cdot A$$

Lemma 2

If $N(x)$ is a polynomial matrix such that $N(x) \cdot (xI - A) = M$, where A and M are matrices (with constant entries), then M must be the zero matrix and $N(x) \equiv 0$.

Proof.

$N(x)$ can be written (see (1))

$$N(x) = N_n x^n + N_{n-1} x^{n-1} + \cdots + N_1 x + N_0$$

$$\longrightarrow M = N_n x^{n+1} + (N_{n-1} - N_n \cdot A) x^n + \cdots + (N_0 - N_1 \cdot A) x - N_0 \cdot A$$

$$\longrightarrow N_n = 0$$

Lemma 2

If $N(x)$ is a polynomial matrix such that $N(x) \cdot (xI - A) = M$, where A and M are matrices (with constant entries), then M must be the zero matrix and $N(x) \equiv 0$.

Proof.

$N(x)$ can be written (see (1))

$$N(x) = N_n x^n + N_{n-1} x^{n-1} + \cdots + N_1 x + N_0$$

$$\longrightarrow M = N_n x^{n+1} + (N_{n-1} - N_n \cdot A) x^n + \cdots + (N_0 - N_1 \cdot A) x - N_0 \cdot A$$

$$\longrightarrow N_n = 0 \longrightarrow N_{n-1} = 0$$

Lemma 2

If $N(x)$ is a polynomial matrix such that $N(x) \cdot (xI - A) = M$, where A and M are matrices (with constant entries), then M must be the zero matrix and $N(x) \equiv 0$.

Proof.

$N(x)$ can be written (see (1))

$$N(x) = N_n x^n + N_{n-1} x^{n-1} + \cdots + N_1 x + N_0$$

$$\longrightarrow M = N_n x^{n+1} + (N_{n-1} - N_n \cdot A) x^n + \cdots + (N_0 - N_1 \cdot A) x - N_0 \cdot A$$

$$\longrightarrow N_n = 0 \longrightarrow N_{n-1} = 0 \longrightarrow \ldots \longrightarrow N_0 = 0$$

Lemma 2

If $N(x)$ is a polynomial matrix such that $N(x) \cdot (xI - A) = M$, where A and M are matrices (with constant entries), then M must be the zero matrix and $N(x) \equiv 0$.

Proof.

$N(x)$ can be written (see (1))

$$N(x) = N_n x^n + N_{n-1} x^{n-1} + \cdots + N_1 x + N_0$$

$$\longrightarrow M = N_n x^{n+1} + (N_{n-1} - N_n \cdot A) x^n + \cdots + (N_0 - N_1 \cdot A) x - N_0 \cdot A$$

$$\longrightarrow N_n = 0 \longrightarrow N_{n-1} = 0 \longrightarrow \ldots \longrightarrow N_0 = 0 \longrightarrow N(x) \equiv 0 \longrightarrow M = 0$$

□

Theorem 2 (again)

The matrix M is a matrix zero of the polynomial $p(x)$ if and only if

$$p(x) \cdot I = R(x) \cdot (xI - M)$$

for some polynomial matrix $R(x)$, where I is the identity matrix.

Theorem 2 (again)

The matrix M is a matrix zero of the polynomial $p(x)$ if and only if

$$p(x) \cdot I = R(x) \cdot (xI - M)$$

for some polynomial matrix $R(x)$, where I is the identity matrix.

Proof.

$(\Longrightarrow)$

Theorem 2 (again)

The matrix M is a matrix zero of the polynomial $p(x)$ if and only if

$$p(x) \cdot I = R(x) \cdot (xI - M)$$

for some polynomial matrix $R(x)$, where I is the identity matrix.

Proof.

$(\Longrightarrow)$ $p(M) = 0$

Theorem 2 (again)

The matrix M is a matrix zero of the polynomial $p(x)$ if and only if

$$p(x) \cdot I = R(x) \cdot (xI - M)$$

for some polynomial matrix $R(x)$, where I is the identity matrix.

Proof.

$(\Longrightarrow)$ $p(M) = 0$

$$x^i I - M^i = (x^{i-1} I + x^{i-2} M + \cdots + xM^{i-2} + M^{i-1}) \cdot (xI - M)$$

Theorem 2 (again)

The matrix M is a matrix zero of the polynomial $p(x)$ if and only if

$$p(x) \cdot I = R(x) \cdot (xI - M)$$

for some polynomial matrix $R(x)$, where I is the identity matrix.

Proof.

$(\Longrightarrow)$ $p(M) = 0$

$$\begin{aligned} x^i I - M^i &= (x^{i-1} I + x^{i-2} M + \cdots + xM^{i-2} + M^{i-1}) \cdot (xI - M) \\ p(x) I - p(M) &= \textstyle\sum_{i=1}^{n} a_i (x^{i-1} I + x^{i-2} M + \cdots + M^{i-1}) \cdot (xI - M) \end{aligned}$$

Theorem 2 (again)

The matrix M is a matrix zero of the polynomial $p(x)$ if and only if

$$p(x) \cdot I = R(x) \cdot (xI - M)$$

for some polynomial matrix $R(x)$, where I is the identity matrix.

Proof.

$(\Longrightarrow)$ $p(M) = 0$

$$\begin{aligned} x^i I - M^i &= (x^{i-1} I + x^{i-2} M + \cdots + x M^{i-2} + M^{i-1}) \cdot (xI - M) \\ p(x) I - p(M) &= \sum_{i=1}^{n} a_i (x^{i-1} I + x^{i-2} M + \cdots + M^{i-1}) \cdot (xI - M) \\ &= Q(x) \cdot (xI - M) \end{aligned}$$

Theorem 2 (again)

The matrix M is a matrix zero of the polynomial $p(x)$ if and only if

$$p(x) \cdot I = R(x) \cdot (xI - M)$$

for some polynomial matrix $R(x)$, where I is the identity matrix.

Proof.

$(\Longrightarrow)$ $p(M) = 0$

$$\begin{aligned} x^i I - M^i &= (x^{i-1} I + x^{i-2} M + \cdots + x M^{i-2} + M^{i-1}) \cdot (xI - M) \\ p(x) I - p(M) &= \textstyle\sum_{i=1}^{n} a_i (x^{i-1} I + x^{i-2} M + \cdots + M^{i-1}) \cdot (xI - M) \\ &= Q(x) \cdot (xI - M) \\ p(x) I &= p(M) + Q(x) \cdot (xI - M) \end{aligned}$$

$$p(x)I = p(M) + Q(x) \cdot (xI - M). \tag{2}$$

$$p(x)I = p(M) + Q(x) \cdot (xI - M). \tag{2}$$

$$p(M) = 0 \longrightarrow p(x)I = Q(x) \cdot (xI - M).$$

$$p(x)I = p(M) + Q(x) \cdot (xI - M). \tag{2}$$

$$p(M) = 0 \longrightarrow p(x)I = Q(x) \cdot (xI - M).$$

$(\Longleftarrow)$

$$p(x)I = p(M) + Q(x) \cdot (xI - M). \tag{2}$$

$$p(M) = 0 \longrightarrow p(x)I = Q(x) \cdot (xI - M).$$

$(\Longleftarrow)$ $p(x)I = R(x) \cdot (xI - M)$

$$p(x)I = p(M) + Q(x) \cdot (xI - M). \tag{2}$$

$$p(M) = 0 \longrightarrow p(x)I = Q(x) \cdot (xI - M).$$

$(\Longleftarrow)$ $p(x)I = R(x) \cdot (xI - M) \wedge (2)$:

$$p(x)I = p(M) + Q(x) \cdot (xI - M). \tag{2}$$

$$p(M) = 0 \longrightarrow p(x)I = Q(x) \cdot (xI - M).$$

$(\Longleftarrow)$ $p(x)I = R(x) \cdot (xI - M) \wedge (2)$:

$$R(x) \cdot (xI - M) = \quad p(M) + Q(x) \cdot (xI - M)$$

$$p(x)I = p(M) + Q(x) \cdot (xI - M). \tag{2}$$

$$p(M) = 0 \longrightarrow p(x)I = Q(x) \cdot (xI - M).$$

$(\Longleftarrow)$ $p(x)I = R(x) \cdot (xI - M) \wedge (2)$:

$$\begin{aligned} R(x) \cdot (xI - M) = &\quad p(M) + Q(x) \cdot (xI - M) \\ p(M) = &\quad \big(R(x) - Q(x)\big) \cdot (xI - M) \end{aligned}$$

$$p(x)I = p(M) + Q(x) \cdot (xI - M). \tag{2}$$

$$p(M) = 0 \longrightarrow p(x)I = Q(x) \cdot (xI - M).$$

($\Longleftarrow$) $p(x)I = R(x) \cdot (xI - M) \wedge (2)$:

$$\begin{aligned} R(x) \cdot (xI - M) &= \quad p(M) + Q(x) \cdot (xI - M) \\ p(M) &= \quad \big(R(x) - Q(x)\big) \cdot (xI - M) \end{aligned}$$

By Lemma 2

$$p(x)I = p(M) + Q(x) \cdot (xI - M). \tag{2}$$

$$p(M) = 0 \longrightarrow p(x)I = Q(x) \cdot (xI - M).$$

($\Longleftarrow$) $p(x)I = R(x) \cdot (xI - M) \wedge (2)$:

$$\begin{aligned} R(x) \cdot (xI - M) &= \quad p(M) + Q(x) \cdot (xI - M) \\ p(M) &= \quad \big(R(x) - Q(x)\big) \cdot (xI - M) \end{aligned}$$

By Lemma 2 $R(x) \equiv Q(x)$ and $p(M) = 0$.

Conclusion

Conclusion

- Our ‘matrix theorem’ is a nice analogue of ‘number theorem’.

Conclusion

- Our 'matrix theorem' is a nice analogue of 'number theorem'.
- Still do not know how to get such a $p(x)$ that $p(M) = 0$ for a given M.

Conclusion

- Our 'matrix theorem' is a nice analogue of 'number theorem'.
- Still do not know how to get such a $p(x)$ that $p(M) = 0$ for a given M.
- No simple answer $\longrightarrow$ complexity of the matrix algebra.

Conclusion

- Our ‘matrix theorem’ is a nice analogue of ‘number theorem’.
- Still do not know how to get such a $p(x)$ that $p(M) = 0$ for a given M.
- No simple answer $\longrightarrow$ complexity of the matrix algebra.
- The unanswered question how to find $p(x)\ldots$ could be nicely addressed later:

Conclusion

- Our ‘matrix theorem’ is a nice analogue of ‘number theorem’.
- Still do not know how to get such a $p(x)$ that $p(M) = 0$ for a given M.
- No simple answer $\longrightarrow$ complexity of the matrix algebra.
- The unanswered question how to find $p(x) \ldots$ could be nicely addressed later:
    - Matrix M is an element of the vector space of $n \times n$ matrices, which has dimension n^2.

Conclusion

- Our ‘matrix theorem’ is a nice analogue of ‘number theorem’.
- Still do not know how to get such a $p(x)$ that $p(M) = 0$ for a given M.
- No simple answer $\longrightarrow$ complexity of the matrix algebra.
- The unanswered question how to find $p(x)\ldots$ could be nicely addressed later:
  - Matrix M is an element of the vector space of $n \times n$ matrices, which has dimension n^2.
  - $I = M^0, M^1, M^2, \ldots, M^{n^2}$ must be a set of linearly dependent vectors...

Conclusion

- Our 'matrix theorem' is a nice analogue of 'number theorem'.
- Still do not know how to get such a $p(x)$ that $p(M) = 0$ for a given M.
- No simple answer $\longrightarrow$ complexity of the matrix algebra.
- The unanswered question how to find $p(x)\ldots$ could be nicely addressed later:
    - Matrix M is an element of the vector space of $n \times n$ matrices, which has dimension n^2.
    - $I = M^0, M^1, M^2, \ldots, M^{n^2}$ must be a set of linearly dependent vectors...
    - So, there must be constants $a_0, a_1, a_2, \ldots, a_{n^2}$ such that

      $a_0 I + a_1 M + a_2 M^2 + \cdots + a_{n^2} M^{n^2} = 0.$

Conclusion

- Our 'matrix theorem' is a nice analogue of 'number theorem'.
- Still do not know how to get such a $p(x)$ that $p(M) = 0$ for a given M.
- No simple answer $\longrightarrow$ complexity of the matrix algebra.
- The unanswered question how to find $p(x) \ldots$ could be nicely addressed later:
    - Matrix M is an element of the vector space of $n \times n$ matrices, which has dimension n^2.
    - $I = M^0, M^1, M^2, \ldots, M^{n^2}$ must be a set of linearly dependent vectors...
    - So, there must be constants $a_0, a_1, a_2, \ldots, a_{n^2}$ such that

      $a_0 I + a_1 M + a_2 M^2 + \cdots + a_{n^2} M^{n^2} = 0.$
    - Therefore, for $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n^2} x^{n^2}$, we have $p(M) = 0$.

Conclusion

- Our 'matrix theorem' is a nice analogue of 'number theorem'.
- Still do not know how to get such a $p(x)$ that $p(M) = 0$ for a given M.
- No simple answer $\longrightarrow$ complexity of the matrix algebra.
- The unanswered question how to find $p(x)\ldots$ could be nicely addressed later:
  - Matrix M is an element of the vector space of $n \times n$ matrices, which has dimension n^2.
  - $I = M^0, M^1, M^2, \ldots, M^{n^2}$ must be a set of linearly dependent vectors...
  - So, there must be constants $a_0, a_1, a_2, \ldots, a_{n^2}$ such that

    $$a_0 I + a_1 M + a_2 M^2 + \cdots + a_{n^2} M^{n^2} = 0.$$
  - Therefore, for $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n^2} x^{n^2}$, we have $p(M) = 0$.
  - With Cayley-Hamilton theorem the statement is refined in the form of proving the existence of such a polynomial of much smaller degree.

Conclusion - Elementary proof of Cayley-Hamilton theorem

- Cayley-Hamilton theorem:

Conclusion - Elementary proof of Cayley-Hamilton theorem

- Cayley-Hamilton theorem: For *characteristic polynomial* $\chi_M(x) = \det(xI - M)$

Conclusion - Elementary proof of Cayley-Hamilton theorem

- Cayley-Hamilton theorem: For *characteristic polynomial* $\chi_M(x) = \det(xI - M)$, we have $\chi_M(M) = 0$.

Conclusion - Elementary proof of Cayley-Hamilton theorem

- Cayley-Hamilton theorem: For *characteristic polynomial* $\chi_M(x) = \det(xI - M)$, we have $\chi_M(M) = 0$.
- To understand characteristic polynomial, one must understand *determinant*.

Conclusion - Elementary proof of Cayley-Hamilton theorem

- Cayley-Hamilton theorem: For *characteristic polynomial* $\chi_M(x) = \det(xI - M)$, we have $\chi_M(M) = 0$.
- To understand characteristic polynomial, one must understand *determinant*.
- To understand *determinant*, one must understand 'row and column Laplace expansion'

Conclusion - Elementary proof of Cayley-Hamilton theorem

- Cayley-Hamilton theorem: For *characteristic polynomial* $\chi_M(x) = \det(xI - M)$, we have $\chi_M(M) = 0$.
- To understand characteristic polynomial, one must understand *determinant*.
- To understand *determinant*, one must understand 'row and column Laplace expansion', which is also an algorithm for obtaining the matrix inverse, if it exists:

$$\det(M) \cdot I = \mathrm{adj}(M) \cdot M. \tag{3}$$

Conclusion - Elementary proof of Cayley-Hamilton theorem

- Cayley-Hamilton theorem: For *characteristic polynomial* $\chi_M(x) = \det(xI - M)$, we have $\chi_M(M) = 0$.
- To understand characteristic polynomial, one must understand *determinant*.
- To understand *determinant*, one must understand 'row and column Laplace expansion', which is also an algorithm for obtaining the matrix inverse, if it exists:

$$\det(M) \cdot I = \mathrm{adj}(M) \cdot M. \tag{3}$$

- If M is any matrix, then $xI - M$ is a rather simple polynomial matrix.

Conclusion - Elementary proof of Cayley-Hamilton theorem

- Cayley-Hamilton theorem: For *characteristic polynomial* $\chi_M(x) = \det(xI - M)$, we have $\chi_M(M) = 0$.
- To understand characteristic polynomial, one must understand *determinant*.
- To understand *determinant*, one must understand 'row and column Laplace expansion', which is also an algorithm for obtaining the matrix inverse, if it exists:

$$\det(M) \cdot I = \mathrm{adj}(M) \cdot M. \tag{3}$$

- If M is any matrix, then $xI - M$ is a rather simple polynomial matrix. Putting $xI - M$ into the equation 3, we get

$$\det(xI - M) \cdot I = \mathrm{adj}(xI - M) \cdot (xI - M).$$

Conclusion - Elementary proof of Cayley-Hamilton theorem

- Cayley-Hamilton theorem: For *characteristic polynomial* $\chi_M(x) = \det(xI - M)$, we have $\chi_M(M) = 0$.
- To understand characteristic polynomial, one must understand *determinant*.
- To understand *determinant*, one must understand 'row and column Laplace expansion', which is also an algorithm for obtaining the matrix inverse, if it exists:

$$\det(M) \cdot I = \mathrm{adj}(M) \cdot M. \tag{3}$$

- If M is any matrix, then $xI - M$ is a rather simple polynomial matrix. Putting $xI - M$ into the equation 3, we get

$$\det(xI - M) \cdot I = \mathrm{adj}(xI - M) \cdot (xI - M).$$

- This equality requires an understanding of the Laplace expansion, which is usually done for matrices with constant entries. It is a straightforward argument that analogous statements hold if elements of the matrix are polynomials.

Conclusion - Elementary proof of Cayley-Hamilton theorem

- Cayley-Hamilton theorem: For *characteristic polynomial* $\chi_M(x) = \det(xI - M)$, we have $\chi_M(M) = 0$.
- To understand characteristic polynomial, one must understand *determinant*.
- To understand *determinant*, one must understand 'row and column Laplace expansion', which is also an algorithm for obtaining the matrix inverse, if it exists:

$$\det(M) \cdot I = \mathrm{adj}(M) \cdot M. \tag{3}$$

- If M is any matrix, then $xI - M$ is a rather simple polynomial matrix. Putting $xI - M$ into the equation 3, we get

$$\det(xI - M) \cdot I = \mathrm{adj}(xI - M) \cdot (xI - M).$$

- This equality requires an understanding of the Laplace expansion, which is usually done for matrices with constant entries. It is a straightforward argument that analogous statements hold if elements of the matrix are polynomials.
- Cayley-Hamilton theorem is a trivial consequence of our theorem.

Appendix 1

Appendix 1

- Standard Cayley-Hamilton theorem proofs are commonly ‘computational’ (for example, Lemma 1.9, p. 426, J. Hefferon, *Linear Algebra*, Orthogonal Publishing L3C, 2017) with essentially elementary reasoning but using advanced and unnecessary specific notations.

Appendix 1

- Standard Cayley-Hamilton theorem proofs are commonly ‘computational’ (for example, Lemma 1.9, p. 426, J. Hefferon, *Linear Algebra*, Orthogonal Publishing L3C, 2017) with essentially elementary reasoning but using advanced and unnecessary specific notations.
- The proof of the Cayley-Hamilton theorem is sometimes given within more abstract setting (for example p. 194 and p. 237, K. Hoffman and R. Kunze, *Linear Algebra*, Prentice Hall, 1971).

Appendix 1

- Standard Cayley-Hamilton theorem proofs are commonly ‘computational’ (for example, Lemma 1.9, p. 426, J. Hefferon, *Linear Algebra*, Orthogonal Publishing L3C, 2017) with essentially elementary reasoning but using advanced and unnecessary specific notations.
- The proof of the Cayley-Hamilton theorem is sometimes given within more abstract setting (for example p. 194 and p. 237, K. Hoffman and R. Kunze, *Linear Algebra*, Prentice Hall, 1971). Also our Theorem 2 and consequentially the Cayley-Hamilton theorem proof could be stated for polynomials and matrices over a ring R and considering associated R-modules instead of vector spaces.

Appendix 1

- Standard Cayley-Hamilton theorem proofs are commonly ‘computational’ (for example, Lemma 1.9, p. 426, J. Hefferon, *Linear Algebra*, Orthogonal Publishing L3C, 2017) with essentially elementary reasoning but using advanced and unnecessary specific notations.
- The proof of the Cayley-Hamilton theorem is sometimes given within more abstract setting (for example p. 194 and p. 237, K. Hoffman and R. Kunze, *Linear Algebra*, Prentice Hall, 1971). Also our Theorem 2 and consequentially the Cayley-Hamilton theorem proof could be stated for polynomials and matrices over a ring R and considering associated R-modules instead of vector spaces.
- Cayley-Hamilton theorem proof is often avoided and only illustrated by examples.

Appendix 1

- Standard Cayley-Hamilton theorem proofs are commonly 'computational' (for example, Lemma 1.9, p. 426, J. Hefferon, *Linear Algebra*, Orthogonal Publishing L3C, 2017) with essentially elementary reasoning but using advanced and unnecessary specific notations.
- The proof of the Cayley-Hamilton theorem is sometimes given within more abstract setting (for example p. 194 and p. 237, K. Hoffman and R. Kunze, *Linear Algebra*, Prentice Hall, 1971). Also our Theorem 2 and consequentially the Cayley-Hamilton theorem proof could be stated for polynomials and matrices over a ring R and considering associated R-modules instead of vector spaces.
- Cayley-Hamilton theorem proof is often avoided and only illustrated by examples. Sometimes the proof is given only for 'diagonalizable matrices'.

Appendix 1

- Standard Cayley-Hamilton theorem proofs are commonly ‘computational’ (for example, Lemma 1.9, p. 426, J. Hefferon, *Linear Algebra*, Orthogonal Publishing L3C, 2017) with essentially elementary reasoning but using advanced and unnecessary specific notations.
- The proof of the Cayley-Hamilton theorem is sometimes given within more abstract setting (for example p. 194 and p. 237, K. Hoffman and R. Kunze, *Linear Algebra*, Prentice Hall, 1971). Also our Theorem 2 and consequentially the Cayley-Hamilton theorem proof could be stated for polynomials and matrices over a ring R and considering associated R-modules instead of vector spaces.
- Cayley-Hamilton theorem proof is often avoided and only illustrated by examples. Sometimes the proof is given only for ‘diagonalizable matrices’.
- At an advanced level ‘the proof for only diagonalizable matrices’ might be connected to *metric space* concepts,

Appendix 1

- Standard Cayley-Hamilton theorem proofs are commonly ‘computational’ (for example, Lemma 1.9, p. 426, J. Hefferon, *Linear Algebra*, Orthogonal Publishing L3C, 2017) with essentially elementary reasoning but using advanced and unnecessary specific notations.
- The proof of the Cayley-Hamilton theorem is sometimes given within more abstract setting (for example p. 194 and p. 237, K. Hoffman and R. Kunze, *Linear Algebra*, Prentice Hall, 1971). Also our Theorem 2 and consequentially the Cayley-Hamilton theorem proof could be stated for polynomials and matrices over a ring R and considering associated R-modules instead of vector spaces.
- Cayley-Hamilton theorem proof is often avoided and only illustrated by examples. Sometimes the proof is given only for ‘diagonalizable matrices’.
- At an advanced level ‘the proof for only diagonalizable matrices’ might be connected to *metric space* concepts, where one could use the fact (for example for matrices over $\mathbb{C}$, which suffice for most purposes) that diagonalizable matrices are dense within the space of all matrices.

Appendix 1

- Standard Cayley-Hamilton theorem proofs are commonly ‘computational’ (for example, Lemma 1.9, p. 426, J. Hefferon, *Linear Algebra*, Orthogonal Publishing L3C, 2017) with essentially elementary reasoning but using advanced and unnecessary specific notations.
- The proof of the Cayley-Hamilton theorem is sometimes given within more abstract setting (for example p. 194 and p. 237, K. Hoffman and R. Kunze, *Linear Algebra*, Prentice Hall, 1971). Also our Theorem 2 and consequentially the Cayley-Hamilton theorem proof could be stated for polynomials and matrices over a ring R and considering associated R-modules instead of vector spaces.
- Cayley-Hamilton theorem proof is often avoided and only illustrated by examples. Sometimes the proof is given only for ‘diagonalizable matrices’.
- At an advanced level ‘the proof for only diagonalizable matrices’ might be connected to *metric space* concepts, where one could use the fact (for example for matrices over $\mathbb{C}$, which suffice for most purposes) that diagonalizable matrices are dense within the space of all matrices. With good understanding of continuity, this suffices to conclude that the Cayley-Hamilton theorem holds for all matrices.

Appendix 2

- Arthur Cayley (1821 - 1895)

Appendix 2

- Arthur Cayley (1821 - 1895) in his original paper (A Memoir on the Theory of Matrices, *Philosophical Transactions of the Royal Society of London*, **148**, 1858, p. 17–37)

Appendix 2

- Arthur Cayley (1821 - 1895) in his original paper (A Memoir on the Theory of Matrices, *Philosophical Transactions of the Royal Society of London*, **148**, 1858, p. 17–37) only proved his theorem for an arbitrary 2×2 matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

Appendix 2

- Arthur Cayley (1821 - 1895) in his original paper (A Memoir on the Theory of Matrices, *Philosophical Transactions of the Royal Society of London*, **148**, 1858, p. 17–37) only proved his theorem for an arbitrary 2×2 matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

in the form of explicitly calculating the expression $M^2 - (a+d)M^1 + (ad - bc)M^0$ to be the zero matrix.

Appendix 2

- Arthur Cayley (1821 - 1895) in his original paper (A Memoir on the Theory of Matrices, *Philosophical Transactions of the Royal Society of London*, **148**, 1858, p. 17–37) only proved his theorem for an arbitrary 2×2 matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

in the form of explicitly calculating the expression $M^2 - (a + d)M^1 + (ad - bc)M^0$ to be the zero matrix. In modern terminology, this corresponds to the statement that $\chi_M(M) = 0$ for $\chi_M(x) = \det(xI - M)$.

Appendix 2

- Arthur Cayley (1821 - 1895) in his original paper (A Memoir on the Theory of Matrices, *Philosophical Transactions of the Royal Society of London*, **148**, 1858, p. 17–37) only proved his theorem for an arbitrary 2×2 matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

in the form of explicitly calculating the expression $M^2 - (a+d)M^1 + (ad - bc)M^0$ to be the zero matrix. In modern terminology, this corresponds to the statement that $\chi_M(M) = 0$ for $\chi_M(x) = \mathtt{det}(xI - M)$. Cayley indicated a similar proof for an arbitrary 3×3 matrix.

Appendix 2

- Arthur Cayley (1821 - 1895) in his original paper (A Memoir on the Theory of Matrices, *Philosophical Transactions of the Royal Society of London*, **148**, 1858, p. 17–37) only proved his theorem for an arbitrary 2×2 matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

in the form of explicitly calculating the expression $M^2 - (a + d)M^1 + (ad - bc)M^0$ to be the zero matrix. In modern terminology, this corresponds to the statement that $\chi_M(M) = 0$ for $\chi_M(x) = \det(xI - M)$. Cayley indicated a similar proof for an arbitrary 3×3 matrix. For matrices of higher dimensions he wrote "*I have not thought it necessary to undertake the labour of a formal proof of the theorem in the general case of a matrix of any degree*".

Appendix 2

- Arthur Cayley (1821 - 1895) in his original paper (A Memoir on the Theory of Matrices, *Philosophical Transactions of the Royal Society of London*, **148**, 1858, p. 17–37) only proved his theorem for an arbitrary 2×2 matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

in the form of explicitly calculating the expression $M^2 - (a + d)M^1 + (ad - bc)M^0$ to be the zero matrix. In modern terminology, this corresponds to the statement that $\chi_M(M) = 0$ for $\chi_M(x) = \det(xI - M)$. Cayley indicated a similar proof for an arbitrary 3×3 matrix. For matrices of higher dimensions he wrote "*I have not thought it necessary to undertake the labour of a formal proof of the theorem in the general case of a matrix of any degree*".

- The theorem was first proved in its general form by Ferdinand Georg Frobenius (1849 - 1917) in his paper Über vertauschbare Matrizen, *Sitz. Preuß. Akad. Wiss. Berlin*, 1896, p. 601–614.

Appendix 2

- Arthur Cayley (1821 - 1895) in his original paper (A Memoir on the Theory of Matrices, *Philosophical Transactions of the Royal Society of London*, **148**, 1858, p. 17–37) only proved his theorem for an arbitrary 2×2 matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

in the form of explicitly calculating the expression $M^2 - (a+d)M^1 + (ad - bc)M^0$ to be the zero matrix. In modern terminology, this corresponds to the statement that $\chi_M(M) = 0$ for $\chi_M(x) = \det(xI - M)$. Cayley indicated a similar proof for an arbitrary 3×3 matrix. For matrices of higher dimensions he wrote "*I have not thought it necessary to undertake the labour of a formal proof of the theorem in the general case of a matrix of any degree*".

- The theorem was first proved in its general form by Ferdinand Georg Frobenius (1849 - 1917) in his paper Über vertauschbare Matrizen, *Sitz. Preuß. Akad. Wiss. Berlin*, 1896, p. 601–614. Our proof somehow resembles Frobenious proof.